APPENDIX

TO THE

MENSURATION.

FOR THE USE OF TEACHERS.

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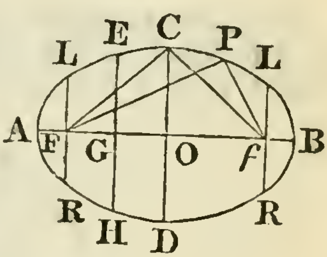
APPENDIX.

PROPERTIES OF THE CONIC SECTIONS.

OF THE ELLIPSIS.

DEFINITIONS.

1. An ellipsis is a plane figure, bounded by a curved line, and is such that if, from any point in the curve, two straight lines be drawn to two certain points, the sum of these lines will always be the same.



Let the pupil fix two pins in a table at any convenient distance, as at F, f; next he is to fasten the two ends of a thread, and throw it loosely over the fixed pins; then by stretching the string with a black-lead pencil or a sharp-pointed instrument, and carrying it gently round, an ellipsis will be formed.

2. The two points, F, f, where the pins are fixed, are called the foci.

3. The line passing through the foci is called the trans-

verse axis, or the axis major. The point O, in the middle of the axis A B, is the centre of the ellipsis.

4. The line C D, drawn through the centre of the ellipsis, perpendicular to the transverse axis A B, is called the conjugate axis, or the axis minor.

5. The line L R, drawn through the focal point F or f, perpendicular to the transverse axis A B, is called the parameter, or latus rectum.

6. A line drawn from any point of the curve, perpendicular to the transverse axis, is called an ordinate to the transverse axis, as E G, or H G. When it goes quite through the ellipsis, as E H, it is called a double ordinate.

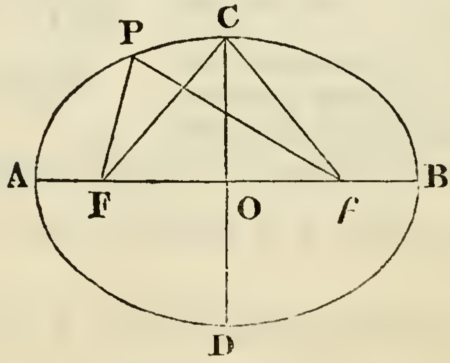
7. The extremity of any diameter is called the vertex; thus, A and B are the vertices of the diameter A B; C and D are the vertices of the diameter C D.

8. That part of the diameter between the vertex and the ordinate is called an abscissa; thus G B and A G are abscissas to the ordinate G E.

PROPOSITION I.

If from any point P, *in an ellipse, straight lines* P F, P f, *be drawn to the foci* F f, *their sum is equal to the transverse axis* A B.

For, from the generation of the curve, it is evident that A f is equal to B F; hence A F $=$ B f. It is plain also that F P $+$ P $f =$ A $f +$ A F $=$ A $f +$ B $f =$ A B.



PROPOSITION II.

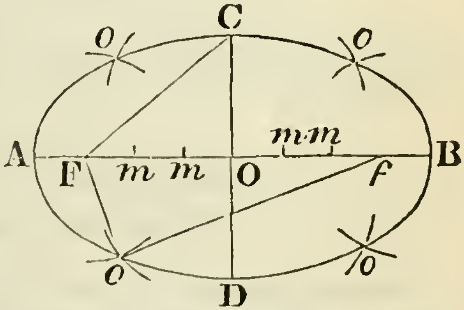
The line connecting the extremity of the conjugate axis and focus of the ellipse is equal to half the transverse axis; that is, $F\,C$ *or* $f\,C = \frac{A\,B}{2} = A\,O$, *or* $O\,B$.

For, by the last Proposition, $F\,C + f\,C = A\,B$; but $F\,C = f\,C$; because in the triangles $F\,o\,C$ and $f\,o\,C$, $F\,o$ is equal to $f\,o$ and $o\,C$ common, and the angles at o right; therefore $F\,C$ is equal to $f\,C$ (4, I.) and hence $F\,C$ is equal to $\frac{A\,B}{2} = A\,O$, or $O\,B$.

PROPOSITION III.

THEOREM.

The transverse and conjugate diameters of an ellipsis being given, to find the foci, and construct the figure.

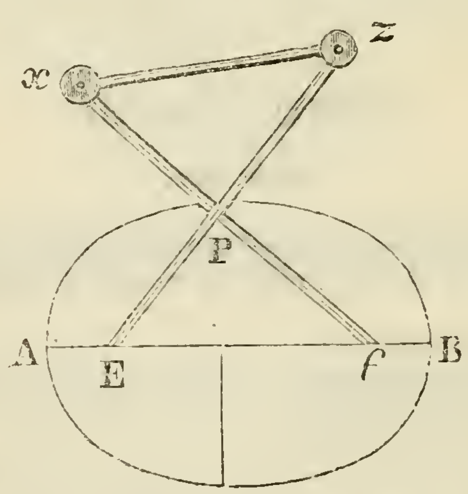


Draw the transverse and conjugate diameters, bisecting each other at right angles at O; from C as a centre, and radius A O, describe an arc cutting the transverse diameter A B in F, *f*, which are the foci of the ellipsis; take a great number of points in A B, the more the better, as *m*, *m*, &c., with the radii $A\,m$, $B\,m$ ($A\,m + B\,m = A\,B$), and with F, *f*, as centres, describe two arcs crossing each other at *o*, *o*, &c. Join *o*, *o*, &c., with the pen, and the curve will be that of an ellipsis.

For, by Proposition II. $F C = f C = A O$: and by Proposition I. $F o + o f = A m + B m = A B$; hence the reason of the construction.

On the same principle, an ellipse may be constructed by means of three rulers.

Provide three rulers, of which two $F z$, $f x$ are equal, each, to the transverse axis A B, and the third $z x$ equal to the focal distance $F f$. Then connecting these rulers so as to move freely about F, f, and also about x, z, their intersection P will always be in the curve; so that if slits run along both rulers, and the instrument be turned freely about the foci, a pencil, or sharp-pointed instrument, introduced through the slits at the point of intersection, will describe an ellipse.

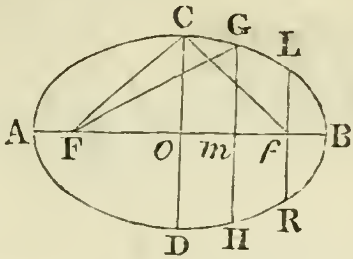


PROPOSITION IV.

The distance between the two foci is a mean proportional between the sum and difference of the transverse and conjugate axis, that is, $A B + C D : F f :: F f : A B - C D$.

For $C o^2 = F C^2 - F o^2 = A o^2 - F o^2$ (Prop. II.) $\therefore 4 C o^2 (= C D^2) = 4 A o^2 - 4 F o^2$; but $4 A o^2 = A B^2$, and $4 F o^2 = F f^2$ (Cor. 4, II.); hence $F f^2 = A B^2$

$-$ C D^2 = (A B + C D) $\times$ (A B — C D) (Cor. 5, II.); therefore A B + C D : Ff :: Ff : B A — C D (17, VI.)



PROPOSITION V.

The square of the distance of the focus from the centre is equal to the difference of the squares of the semi-axes; that is, F o^2 = A o^2 — C o^2.

For, F o^2 = F C^2 — o C^2 (47, I.); but F C = A o (Prop. II.); therefore F o^2 = A o^2 — o C^2.—See the last figure.

PROPOSITION VI.

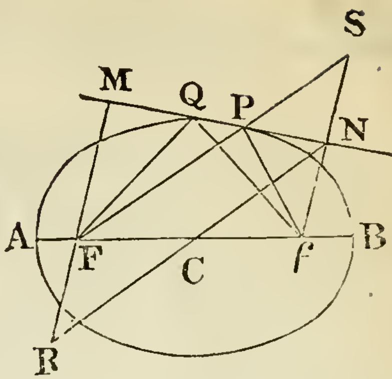
The rectangle of the distances of either focus from both vertices is equal to the square of the semi-conjugate; that is, A F $\times$ F B = C o^2.

Because C o^2 = F C^2 — F o^2; and F C = A o; therefore C o^2 = A o^2 — F o^2 = A F $\times$ F B (Cor. 5, II.)—See the last figure.

PROPOSITION VII.

To draw a tangent to an ellipsis, from a given point P *in the curve.*

Join F P, f P; bisect the exterior angle f P S. The bisecting line P N will be a tangent.



For, if not, it will, if produced, cut the curve in some other point, suppose Q. Make P S = P f; join f S, S Q,* Q f, Q F: then (4, I.), f N = N S, and the angle f N Q = S N Q ∴ (4, I.) Q S = Q f ∴ F Q + Q S = (F Q + Q f = F P + P f) = F S, which is impossible (20, I.): therefore, P N is a tangent to the curve.

Cor. 1. The line joining N and the centre C is equal to half F S (4, VI.) because it bisects f F and f S; but F S = A B ∴ C N = A C. In like manner, it may be proved that if F M be perpendicular to the tangent, C M = A C.

Cor. 2. Produce N C, M F, to meet in R; then (26, I.), C R = C N, and F R = f N; therefore, since R M N is a right angle, a circle having C for its centre, and radius C A, will pass through the points A M N B R, ∴ M F × F R = M F × f N = A F × F B = the square of the semi-conjugate. (Prop. VI.)

PROPOSITION VIII.

The square of half the transverse axis is to the square of half the conjugate, as the rectangle of any two abscissas to the square of the ordinate which divides them; that is, $A c^2 : c D^2 :: A k \times k B : m k^2$.

To understand this, it is necessary to premise the following lemma, viz.

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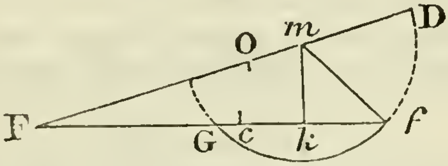
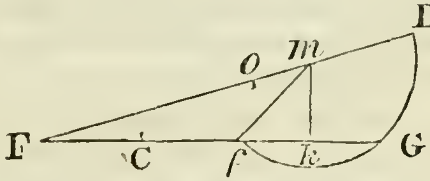
* Q S may be drawn with the pen.

If from the vertex of any plane triangle a perpendicular be let fall on the base, or base produced, and the base be bisected; then half the base is to half the sum of the sides, as the difference between half the sum of the sides and one of them is to the distance between the middle of the base and perpendicular.

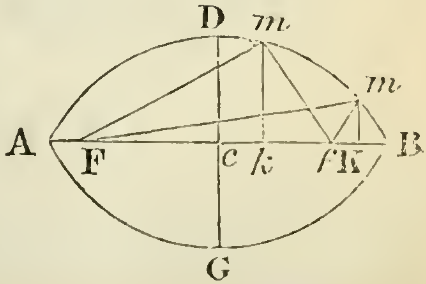
Make F O = half the sum of the sides F m, f m; then it will be F c : F O :: O m or F O — f m : c k.





Because F f : F D :: F m — f m : F G. Hence F c : F O :: O m : $\frac{1}{2}$ F G = $\frac{1}{2}$ F f $\pm$ $\frac{1}{2}$ f G = f c $\pm$ f k = c k.

By the lemma A c : F c :: c k : A c — f m, which being squared and divided, will be A c^2 : A c^2 — F c^2 :: c k^2 : c k^2 — A c^2 + 2 A c $\times$ f m — f m^2; then, by alternation and conversion, A c^2 : A c^2 — F c^2 :: A c^2 — c k^2 : 2 A c^2 — F c^2 — c k^2 — 2 A c $\times$ f m + f m^2; but A c^2 — F c^2 = c D^2, also A c^2 — c k^2 = A k $\times$ k B, and 2 A c^2 — 2 A c $\times$ f m = 2 F c $\times$ c k; likewise f m^2 — F c^2 — c k^2 + 2 F c $\times$ c k = f m^2 — f k^2 = m k^2 $\therefore$ A c^2 : c D^2 :: A k $\times$ k B : m k^2.



Cor. 1. The latus rectum is a third proportional to the axis major and axis minor.

For $A\,c^2 : c\,D^2 :: A\,F \times F\,B$ or $c\,D^2 : F\,L^2$* $\therefore A\,c : c\,D :: c\,D : F\,L \therefore 2\,A\,c : 2\,c\,D :: 2\,c\,D : 2\,F\,L$.

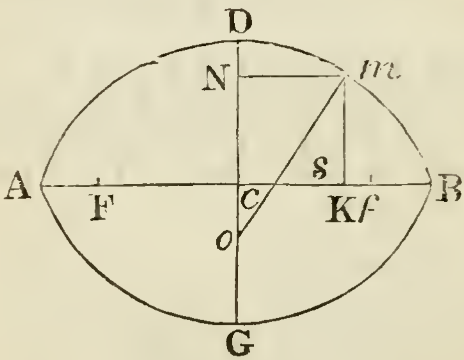
Cor. 2. The transverse axis is to the latus rectum, or parameter, as the rectangle or any two abscissas is to the square of the ordinate which divides them. Because, by Cor. 1, $D\,G^2 = A\,B \times p$ (putting p for the parameter); therefore $A\,B^2 : A\,B \times p :: A\,k \times k\,B : m\,k^2$; hence $A\,B : p :: A\,k \times k\,B : m\,k^2$.

Cor. 3. Hence the rectangles of every pair of abscissas are proportional to the squares of their corresponding ordinates.

Cor. 4. The square of half the conjugate is to the square of half the transverse, as the rectangle of any two abscissas of the conjugate is to the square of the corresponding ordinate.

For, by the Prop. $A\,c^2 : c\,D^2 :: A\,c^2 - c\,K^2 : m\,K^2$ or $c\,N^2$; then, by inversion and division, $c\,D^2 : A\,c^2 :: (c\,D^2 - c\,N^2 =)\ D\,N \times N\,G : c\,K^2 = m\,N^2$.—See next figure.

Cor. 5. Hence also the rectangles of the abscissas of the conjugate are proportional to the squares of the ordinates which divide them.



* F L, which here signifies the latus rectum, is not in the figure, but may be drawn with the pen.—See Definition 5.

Cor. 6. As $A\,c^2$ ($= F\,D^2$) $= c\,D^2 + F\,c^2$; then $c\,D^2 + F\,c^2 : c\,D^2 :: A\,K \times K\,B : m\,K^2$.

Cor. 7. By Cor. 4, $c\,D^2 : c\,D^2 + F\,c^2 :: D\,N \times N\,G : N\,m^2$.

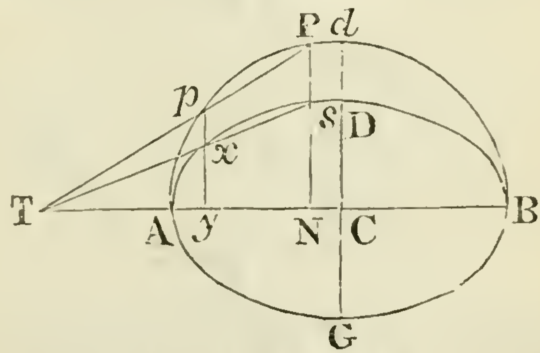
Cor. 8. As $c\,D^2 = A\,c^2 - F\,c^2$; then, by Cor. 4, $A\,c^2 - F\,c^2 : A\,c^2 :: D\,N \times N\,G : N\,m^2$.

Cor. 9. Because $A\,B = \dfrac{D\,G^2}{p}$; therefore, by Cor. 2, $D\,G^2 : p^2 :: A\,K \times K\,B : m\,K^2$.

Cor. 10. From m draw $m\,o =$ half the transverse, then will $m\,s =$ half the conjugate. For, by similar triangles, $m\,o^2$ or $A\,c^2 : m\,N^2$, or $c\,K^2 :: s\,m^2 : s\,K^2$; then, by alternation and division, $A\,c^2 : s\,m^2 :: A\,c^2 - c\,K^2 : (s\,m^2 - s\,K^2 =)\ m\,K^2 \therefore s\,m = C\,D$.

Cor. 11. Therefore, if from a point o be laid off $o\,s =$ half the difference between the diameters, and that line be produced till it becomes equal to half the transverse, its extremity will be in the curve.

Cor. 12. If upon either axis a circle be described, the corresponding circular and elliptic ordinates will be proportional. Because the rectangles of the abscissas are equal to the squares of the circular ordinates. (35, III.)



Cor. 13. If through the extremities of two unequal elliptic ordinates a straight line be drawn, so as to cut the axis produced, a straight line drawn from the point of intersection will pass through the extremities of the corresponding cir-

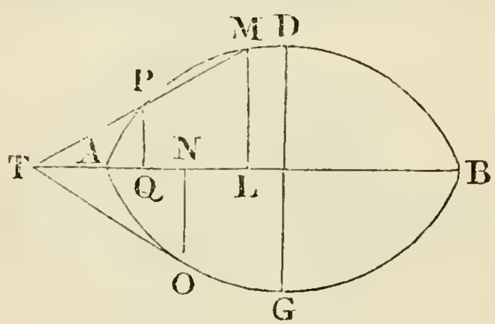
cular ordinates. Because $PN : py :: SN : xy$ (Cor. 12); but $SN : xy :: TN : Ty \therefore PN : py :: TN : Ty$.

Cor. 14. From this we may infer that tangents drawn to any two corresponding ordinates will pass through the same point in the axis.

PROPOSITION IX.

If through the extremities of two unequal ordinates a right line be drawn, so as to cut the axis produced; then, as the square of the distance of one of these ordinates from the point of intersection, is to the rectangle of the abscissas, which it divides, so is the sum of the distances of the ordinates from the point of intersection, to the sum or difference of the distances of the ordinates from the centre, according as they fall on the same or contrary sides of the centre; that is, $TQ^2 : AQ \times QB :: TQ + TL : cQ \pm cL$.

From the property of the curve and similar triangles $TQ^2 : AQ \times QB :: TL^2 : AL \times LB$, and therefore, $TQ^2 : AQ + QB :: TL^2 - TQ^2 : AL \times LB - AQ \times$



QB. Now, $TL^2 - TQ^2 = (TL + TQ) \times (TL - TQ)$; but $TL = Tc \mp cL$, and $TQ = Tc - cQ \therefore TL + TQ = 2Tc - cQ \mp cL$, and $TL - TQ = cQ \mp cL$ $\therefore TL^2 - TQ^2 = 2Tc \times cQ \mp 2Tc \times cL + cL^2 - cQ^2$; and $AL \times LB = Ac^2 - cL^2$, BL being the sum of

$A\,c$, $c\,L$, and $A\,L$ their difference, in like manner, $A\,Q \times Q\,B = A\,c^2 - c\,Q^2 \therefore A\,L \times L\,B - A\,Q \times Q\,B = c\,Q^2 - c\,L^2 \therefore T\,Q^2 : A\,Q \times Q\,B :: 2\,T\,c \times c\,Q \mp 2\,T\,c \times c\,L + c\,L^2 - c\,Q^2 : c\,Q^2 - c\,L^2$; which divided by $c\,Q \mp c\,L$; $T\,Q^2 : A\,Q \times Q\,B :: (2\,T\,c - c\,Q + c\,L : c\,Q \pm c\,L ::)\,T\,Q + T\,L : c\,Q \pm c\,L$.

Cor. 1. When Q and L coincide, $T\,M$ will become a tangent, and $c\,Q$, $c\,L$ will each become equal to $c\,N$; therefore $T\,N^2 : A\,N \times N\,B :: 2\,T\,N : 2\,c\,N$; that is, $c\,N : B\,N :: A\,N : T\,N \therefore c\,N \times T\,N = A\,N \times B\,N$.

Cor. 2. From the last analogy, we get $c\,N : B\,N - c\,N :: A\,N : T\,N - A\,N$; $\therefore c\,N : A\,c :: A\,N : A\,T$.

Cor. 3. By alternation and composition, the first analogy becomes $c\,N : c\,N + A\,N :: B\,N : B\,N + T\,N$; that is, $c\,N : A\,c :: B\,N : B\,T$.

Cor. 4. From the two last corollaries, we get $A\,N : B\,N :: A\,T : B\,T$.

Cor. 5. By alternation and composition, the second corollary becomes $c\,N : c\,N + A\,N :: A\,c : A\,c + A\,T$; that is, $c\,N : c\,A :: c\,A : c\,T$.

Cor. 6. By inversion and composition, the last becomes $c\,T : c\,A :: c\,T + c\,A : c\,A + c\,N$; that is, $c\,T : c\,A :: B\,T : B\,N$.

Cor. 7. From the fifth, we get $c\,T : c\,A :: c\,T - c\,A : c\,A - c\,N$, that is, $c\,T : c\,A :: A\,T : N\,A$.

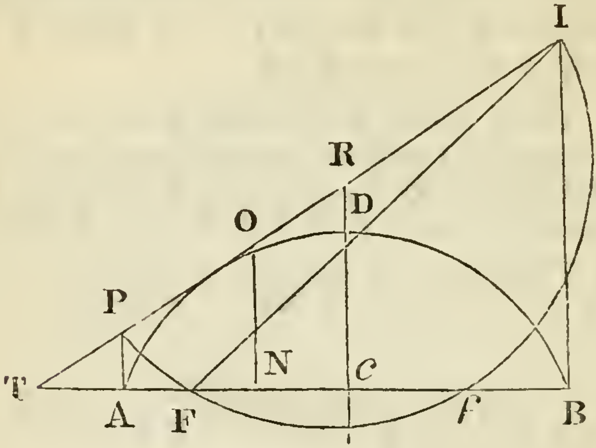
Cor. 8. From the sixth, $c\,T : B\,T :: c\,T - A\,c : B\,T - B\,N$; that is, $c\,T : B\,T :: A\,T : N\,T$.

Cor. 9. By the sixth and eighth, $c\,A : B\,N :: A\,T : N\,T$.

Cor. 10. By the seventh and eighth, $c\,A : B\,T :: A\,N : N\,T$.

Cor. 11. If tangents be drawn at each vertex of the curve, and if the conjugate be produced to meet any other tangent, then as one of these tangents is to the ordinate

drawn from the point of contact, so is the semi-conjugate produced to the other tangent. For (Cor. 8) c T : B T :: A T : N T ∴ C R : B I :: A P : N O; hence, B I : N O :: c R : A P.



Cor. 12. By similar triangles, T B : T N :: B I : N O ∴ by the 10th Cor. A C : A N :: B I : O N.

Cor. 13. By similar triangles, T A : T N :: A P : N O; therefore, by the 9th Cor. A C : B N :: A P : N O.

Cor. 14. The 12th and 13th Ex. Equo. perturbate A N : B N :: A P : B I.

Cor. 15. By compounding the 12th and 13th, A C^2 : A N × N B :: A P × B I : O N^2; that is, C D^2 : N O^2 :: A P × B I : N O^2 (Prop. VII. Cor. 1); therefore C D^2 = A P × B I; that is, the semi-conjugate axis is a mean proportion between the vertical tangents.

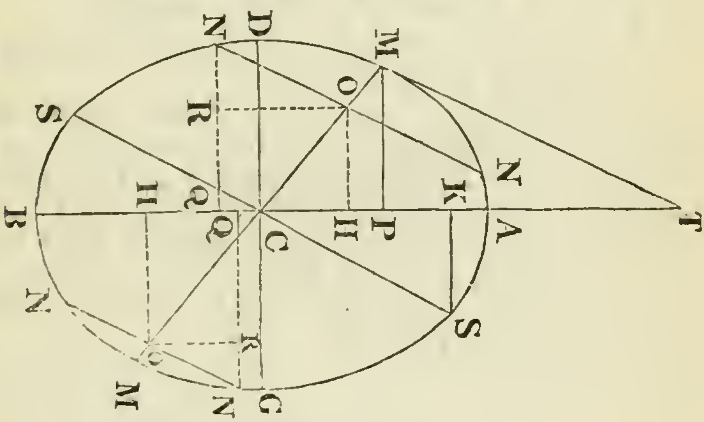
Cor. 16. Hence A P × B I (= C D^2) = A F × F B = A f × f B; therefore, A F : A P :: B I : F B, and A f : A P :: B I : f B; therefore the triangles are similar and right, and the triangles P F I and P f I are right-angled at F and f;* therefore R is the centre of a circle which will pass through the points P, F, f, and I.

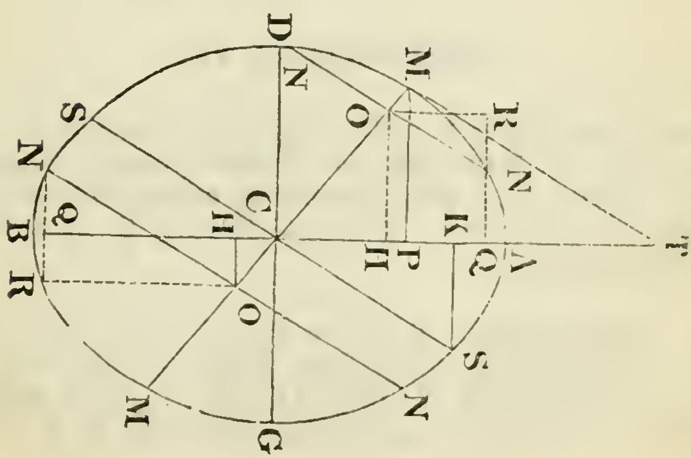
* The lines P F, P f, I f, which are not in the cut, may be drawn with the pen.

PROPOSITION X.

In the ellipse, the square of any diameter is to the square of its conjugate, as the rectangle of the abscissas to that diameter to the square of the ordinate which divides them; that is, $CM^2 : CS^2 :: MO \times OM : ON^2$.

From the point O draw O R parallel to the axis A B, and O H perpendicular to it: also through the point N, the extremity of the ordinate, draw R Q parallel to D C, and draw the ordinates M P and K S.





Then by similar triangles $N Q = \left(\frac{C H}{C P} \backsim \frac{O R}{T P}\right) \times M P$; hence, $(C P \times P T)^2 : M P^2 :: C H^2 \times T P^2 \mp 2 C H \times O R \times C P \times P T + O R^2 \times C P^2 : N Q^2$, but $A P \times P B : M P^2 :: A C^2 - C H^2 \mp 2 C H \times O R - O R^2 : N Q^2$ (Prop. VIII.): and $A P \times P B = C P \times P T$ (Prop. IX. Cor. I.); therefore, $C H^2 \times T P^2 + 2 C H \times O R \times C P \times P T + O R^2 \times C P^2 = (A C^2 - C H^2 \mp 2 C H \times O R - O R^2) \times C P \times P T$; hence $O R^2 = \frac{A C^2 \times C P \times P T - C H^2 \times (C P \times P T + T P^2)}{C P \times P T + C P^2}$ but $A C^2 = T C \times C P = C P \times P T + C P^2$; and $C P \times P T + T P^2 = C T \times T P$; therefore $\frac{P T \times T C \times C P^2 - P T \times T C \times C H^2}{T C \times C P} = O R^2$; that is, $C P : P T :: C P^2 - C H^2 : O R^2$; but by similar triangles, $C S^2 : C K^2 :: N O^2 : O R^2$; that is, $C P \times P T : C S^2 :: O R^2 : O N^2$ (for $C K^2 = A P \times P B = C P \times P T$); hence, by compounding, $C P^2 : C S^2 :: C P^2 - C H^2 : O N^2$; but by similar triangles and division, $C P^2 : C H^2 :: C M^2 : C O^2$, and $C P^2 : C M^2 :: C P^2 - C H^2 : C M^2 - C O^2$; therefore, Ex. Equo. $C M^2 : C S^2 :: C M^2 - C O (= M O \times O M) : N O^2$.

PROPOSITION XI.

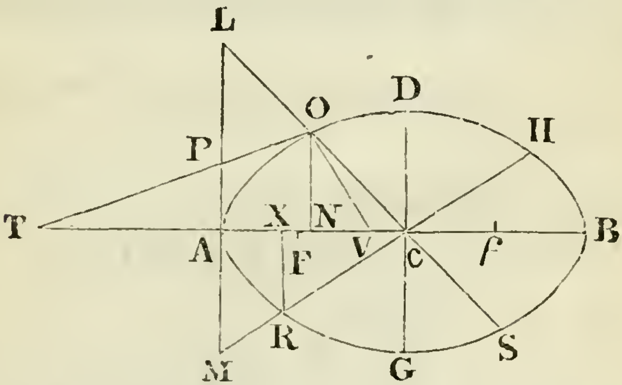
If from a vertex of each of two conjugate diameters an ordinate be drawn to the axis, the distance from one ordinate to the centre is a mean proportion between the abscissas of the other; that is, $c X^2 = A N \times N B$, *or* $c N^2 = A X \times X B$.

By similar triangles, $T N^2 : c X^2 :: [N O^2 : R X^2 ::] A N \times N B : A c^2 - c X^2$ (Prop. VIII. Cor. 3). But $A N \times N B = T N \times N C$ (Prop. IX. Cor. 1); and by Prop. IX. Cor. 5, $A c^2 = T c \times c N$; therefore, $T N : c X^2$

$:: N\,c : T\,c \times c\,N - c\,X^2$, and by composition, $T\,N : T\,c :: c\,X^2 : T\,c \times c\,N$; that is, $c\,X^2 = T\,N \times N\,c$ $=$ (Prop. IX. Cor. 1) $A\,N \times N\,B$.

Cor. 1. Since $c\,X^2 = A\,N \times N\,B = A\,c^2 - c\,N^2$, and $c\,N^2 = A\,X \times X\,B = A\,c^2 - c\,X^2 \therefore c\,X^2 + c\,N^2 = 2\,A\,c^2 - (c\,N^2 \times c\,X^2) \therefore c\,A^2 = c\,X^2 + c\,N^2$.

Cor. 2. By the 35, III. $A\,N \times N\,B =$ the square of the ordinate of the circle applied to the point N; therefore the distance between the centre and one of the ordinates is equal to the other ordinate produced to the circle.



Cor. 3. Hence (and Prop. VIII. Cor. 12) $A\,c : c\,D :: c\,X : N\,O$. Likewise, $A\,c : c\,D :: c\,N : R\,X$; therefore, $A\,c^2 : c\,D^2 :: N\,c \times c\,X : R\,X \times N\,O$.

Cor. 4. Produce the conjugate diameters till they meet a tangent drawn through a vertex of the curve; then, by similar triangles, $c\,N : N\,O :: A\,c : A\,L$; and $c\,X : X\,R :: A\,c : A\,M$: therefore, $N\,c \times c\,X : R\,X \times N\,O :: A\,c^2 : L\,A \times A\,M$; then, by Cor. 2, $c\,D^2 = L\,A \times A\,M$.

Cor. 5. Since, by the third, $A\,c^2 : c\,D^2 :: c\,X^2 : N\,O^2$ and $A\,c^2 : c\,D^2 :: c\,N^2 : R\,X^2$; therefore $A\,c^2 : c\,D^2 :: c\,X^2 + c\,N^2 : N\,O^2 + R\,X^2$.

But, by Cor. 1, $c\,X^2 + c\,N^2 = A\,c^2$; therefore, $c\,D^2 = N\,O^2 + R\,X^2$; that is, the square of half the conjugate is equal to the sum of the squares of the ordinates drawn from the vertices of two semi-conjugates to the axis.

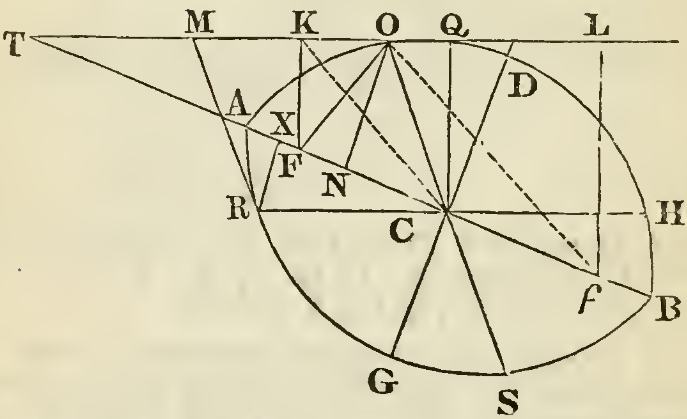
Cor. 6. From this it appears that the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the transverse and conjugate diameters. For $A\,c^2 + c\,D^2 = (N\,O^2 + R\,X^2 + c\,X^2 + c\,N^2 =)\; c\,O^2 + c\,R^2$.

Cor. 7. Hence the sum of the squares of any two conjugate diameters is equal to the sum of the squares of any other two conjugates.

PROPOSITION XII.

A parallelogram described about any two conjugate diameters, is equal to that described about the transverse and conjugate.

Draw the perpendicular C Q; then, by similar triangles, T C : C Q :: C R : R X. But C A : C T :: C N : C A



(Prop. IX. Cor. 5), and C N : R X :: A C : C D (Prop. XI. Cor. 3); therefore, by compounding, C A : C Q :: C R : D C; hence, $A\,C \times C\,D = C\,Q \times C\,R$.

Cor. 1. Therefore, all parellelograms described about the conjugate diameters of an ellipsis are equal.

Cor. 2. By similar triangles, C K or A C* : C Q :: f O : f L; therefore, by the Proposition, f O : f L :: C R :

* C K = A C, by Cor. 1, Proposition VII.

C D; also F O : F K :: C R : C D; that is, as the distance of the focus from the point of contact, is to the perpendicular from the focus to the tangent, so is the conjugate diameter parallel to the tangent, to the conjugate axis.

Cor. 3. Hence f L is as $\frac{f\,O}{C\,R}$, and F K as $\frac{F\,O}{C\,R}$.

Cor. 4. By compounding the two last analogies of the 2nd corollary, F O $\times$ O f : f L $\times$ F K :: C R^2 : C D^2. But C $D^2 = f$ L $\times$ F K (Cor. 2, Prop. VII.); therefore, F O $\times$ Of = C R^2; that is, the semi-conjugate diameter, parallel to any tangent, is a mean proportional between the distances of the foci from the point of contact.

PROPOSITION XIII.

If upon either axis of the ellipsis a circle be described, the area of the circle will be to that of the ellipsis, as the axis upon which the circle was described, to its conjugate.

By Prop. VIII. Cor. 12, the circular is to the corresponding elliptic ordinate, as the axis upon which the circle is described, to its conjugate axis; therefore, the sum of all the circular ordinates, or the area of the circle, is to the sum of all the elliptic ordinates, or the area of the ellipsis, as the diameter upon which the circle is described, to the conjugate diameter.

Cor. 1. Therefore, the ellipsis is a mean proportional between the circumscribed and inscribed circles.

Cor. 2. Hence, also, a circle whose diameter is a mean proportional between the axes is equal to the ellipsis.

Cor. 3. From this Proposition it appears that all ellipses are as their circumscribing parallelograms.

Cor. 4. The area of any circular segment is to the area of the corresponding elliptical segment, as the transverse to the conjugate, or generally as the diameter upon which the circle is described, to its conjugate diameter.

PROPOSITION XIV.

The sphere is to the inscribed spheroid as the square of the transverse to the square of the conjugate.*

This follows from the nature of the circle and ellipsis. Because the areas of any two corresponding circles in each, are as the squares of the diameters; that is, as the square of the transverse to the square of the conjugate; therefore, the sum of all the spherical circles, is to the sum of all the elliptical circles, as the square of the transverse to the square of the conjugate; but the sum of all the spherical circles is the sphere, and the sum of all the elliptic circles is the spheroid, therefore, the sphere is to the spheroid, as the square of the transverse to the square of the conjugate.

Cor. 1. The cylinder circumscribing the sphere is to that circumscribing the spheroid, as the square of the transverse to the square of the conjugate; therefore, the spheroid is two-thirds of the circumscribing cylinder.

Cor. 2. Let t and c be the transverse and conjugate, and $n = \cdot 7854$; $\frac{2\,c^2\,t\,n}{3}$ is the solidity or volume of the sphere.

Cor. 3. The corresponding segments of the sphere and spheroid, are as the square of the transverse to the square of the conjugate, and consequently, in the same ratio with the solids themselves.

Cor. 4. The spheroid is to the inscribed sphere, as the transverse to the conjugate. Because their circumscribing cylinders are as the transverse to the conjugate.

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* A spheroid is a solid, generated by the rotation of a semi-ellipsis about one of its axis, which remains fixed. When the ellipsis revolves about the transverse axis, the figure is called a prolate spheroid, which resembles an egg; when the ellipsis revolves about the shorter axis, the figure is called an oblate spheroid, which resembles an orange.

PROPOSITION XV.

The oblate spheroid is to the inscribed sphere, as the square of the transverse is to the square of the conjugate.

Because the corresponding circles in both solids are as the square of the transverse to the square of the conjugate; therefore, the sum of all the corresponding circles must be in that ratio; that is, the oblate spheroid is to the inscribed sphere as the square of the transverse is to the square of the conjugate.

Cor. 1. The oblate spheroid is two-thirds of the circumscribing cylinder; therefore the volume of the oblate spheroid is equal to $\frac{2\ n\ t^2\ c}{3}$.

Cor. 2. The corresponding segments of the spheroid and inscribed sphere are as the square of the transverse to the square of the conjugate.

Cor. 3. The oblate spheroid is to the circumscribed sphere, as the conjugate to the transverse. Because their circumscribing cylinders are in the ratio of the conjugate to the transverse.

Cor. 4. If about the two axes of an ellipse there be generated two spheres, and two spheroids, the four solids will be continued proportionals; and the common ratio will be that of the two axes of the ellipse; that is, as the sphere upon the greater axis is to the oblate spheroid, so is the oblate spheroid to the prolate spheroid, and as the oblate spheroid is to the prolate spheroid, so is the prolate spheroid to the less sphere, and so is the transverse to the conjugate.

Cor. 5. The oblate spheroid is to the inscribed sphere, as the circumscribed sphere to the prolate spheroid.

Cor. 6. From Cor. 4, the prolate spheroid is a mean proportional between the oblate spheroid and the inscribed sphere.

Cor. 7. And also, the oblate spheroid is a mean proportional between the prolate spheroid and the circumscribed sphere.

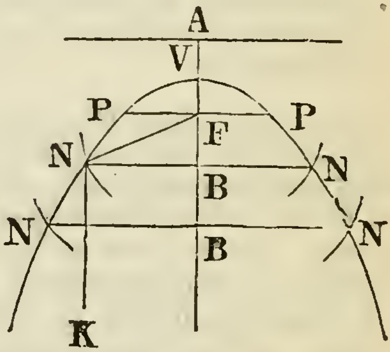
OF THE PARABOLA.

DEFINITIONS.

1. If in an indefinite right line A B, any two points A and F be assumed, and from the centre F, with a radius equal to any distance, A B, a circle be described intersecting in the points, N N, a perpendicular to A B, drawn through the point B, and an infinite number of such points be found, the curve passing through them will be that of a parabola.

2. The point F is called the *focus*.

3. The indefinite line A B is called the *axis;* and the point V, in which the curve intersects it, is called the *principal vertex.*



4. A right line drawn from any point in the curve parallel to the axis, is called a diameter, as N K, and the point in the curve, from which it is drawn, is called its *vertex.*

5. A right line drawn from the curve to any diameter, parallel to a tangent at its vertex is called an ordinate.

6. An ordinate which passes through the focus, and is produced to meet the curve, is called the *parameter* of its diameter.

7. The distance between the vertex of any diameter and the intersection of an ordinate, is called an abscissa.

8. The parameter to the axis A B is called the *latus rectum*.

9. A line drawn at right angles to the axis at A is called the *directrix*.

PROPOSITION I.

The latus rectum P P, *is four times the focal distance from the vertex.*

It is evident from Definition 1, that the vertex V bisects A F; that is, $F V = V A$; and from the same Definition, $F A = F P$ $\therefore$ $2 F V = F P$; therefore $P P = 4 F V$.

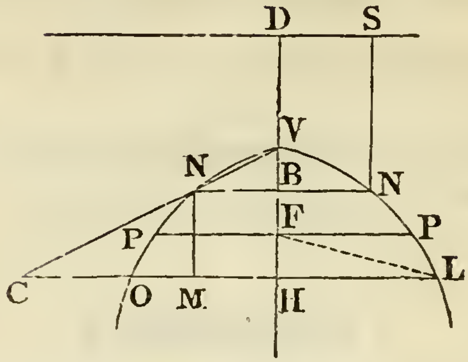
PROPOSITION II.

The sum of any abcissa and focal distance is equal to the distance of the focus from the extremity of the ordinate; that is, $F N = V B + V F$.

Because $F N = A B = V B + V A = V B + V F$.

PROPOSITION III.

Every ordinate to the axis is a mean proportional between its abscissa and the latus rectum; that is, $4 V F \times B V = B N^2$.



☞ Because $F\ N^2 - F\ B^2 = B\ N^2 = (F\ N + F\ B) \times (F\ N - F\ B)$.

Hence, $(F\ N - F\ B) : B\ N :: B\ N : (F\ N + F\ B)$. But (Prop. II.) $F\ N = V\ B + V\ F$; and $F\ B = V\ F \backsim V\ B$; therefore, $F\ N - F\ B = 2\ V\ B$, or $2\ V\ F$; and also, $F\ N + F\ B = 2\ V\ F$, or $2\ V\ B$. Hence, $2\ V\ B : B\ N :: B\ N : 2\ V\ F$, and then, $B\ N^2 = 4\ V\ F \times V\ B$.

Cor. 1. The squares of the ordinates are proportional to their abscissas; since $4\ V\ F$ is a constant quantity.

Cor. 2. Hence the equation of the curve is $y^2 = p\ x$, p, x and y, being the latus rectum, abscissa, and ordinate.

PROPOSITION IV.

The latus rectum [P] *is to the sum of any two ordinates, as their difference is to the difference of the abscissas.*

For $P = 4\ P\ F$ [Prop. I.]; therefore, $P \times V\ B = B\ N^2$ and $P \times V\ H = L\ H^2$ [Prop. III.]; hence, $P \times B\ H$, or $P \times M\ N = L\ H^2 - B\ N^2$; therefore $P : [L\ H + B\ N] :: [L\ H - B\ N] : M\ N$ that is, $P : L\ M :: M\ O : M\ N$

PROPOSITION V.

Any abscissa is to the square of its ordinate, as any right line drawn within the curve parallel to the axis, is to the rectangle of the parts of the double ordinate, which it divides.

For, by Proposition IV. $P \times M N = L M \times M O$, and $P \times V B = B N^2$; hence, $V B : B N^2 :: M N : L M \times M O$.

Cor. The difference between any two abscissas is directly proportional to the rectangle under the sum and difference of their corresponding ordinates.

PROPOSITION VI.

If from the vertex a right line be drawn through the extremity of an ordinate, so as to meet another ordinate produced, that other ordinate will be a mean proportional between itself produced and the first ordinate; that is, $H O^2 = H C \times B N$.

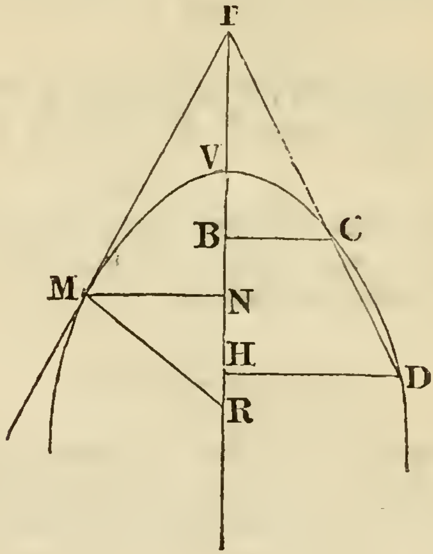
For, by similar triangles, $V B : V H :: B N : H C :: B N^2 : H O^2$ (Prop. III. Cor. 1); therefore, $H O^2 = H C \times B N$.

Cor. The abscissa of the produced ordinate is to the produced ordinate, as the other ordinate to the parameter, or latus rectum. Because $H O^2 = P \times V H = H C \times B N$; therefore, $V H : H C :: B N : P$.

PROPOSITION VII.

If through the extremities of any two ordinates, a right line be drawn so as to cut the axis; the external part of the axis will be a mean proportional between the abscissas; that is, $T V^2 = V H \times V B$.

By similar triangles $T B^2 : T H^2 :: B C^2 : H D^2$; but $B C^2 : H D^2 :: V B : V H$ (Prop. III. Cor.); therefore, $T V^2 + 2 T V \times V B + V B^2 : T V^2 + 2 T V \times V H + V H^2 :: V B : V H$ (4, II.); but by division, $T V^2 \times$



$2 T V \times V B + V B^2 : 2 T V \times V H + V H^2 - 2 T V \times V B - V B^2 :: V B : V H - V B$, and by dividing the second and fourth terms by $V H - V B$, we get

$T V^2 + 2 T V \times V B + V B^2 : 2 T V + V B + V H :: V B : 1$; therefore, $T V^2 + 2 T V \times V B + V B^2 = 2 T V \times V B + V B^2 + V B \times V H$; hence, $T V^2 = V B \times V H$.

Cor. 1. Since $V B = \frac{B C^2}{P}$, and $V H = \frac{B C^2}{P}$ (Prop. III.); therefore, $V B \times V H = \frac{B C^2}{P} \times \frac{H D^2}{P}$; hence, $T V^2 = \frac{B C^2 \times H D^2}{P^2}$; then, $T V = \frac{B C \times H D}{P}$; that is, as the parameter is to one of the ordinates, so is the other ordinate to the external part of the axis.

Cor. 2. When D and C coincide, then T D will be a tangent, as T M, and the abscissas will be equal to each other, and to V N; therefore, as $TV^2 = VB \times VH = VN^2$ (for H and B will coincide at N): hence, $TV = VN^2$; that is, the abscissa will be equal to the external part of the axis.

Definition. If from the point of contact M, M R be drawn at right angles to T M, N R is called the sub-normal.

Cor. 3. The sub-normal is equal to half the parameter.

Because, (8, VI.) $NR = \dfrac{NM^2}{2VN} = \frac{1}{2}P$, (Prop. III.)

Cor. 4. The focus is equi-distant from the point of contact and the intersection of the tangent with the axis.

Because, $FM = VF + VN = TF$. See next figure.

Cor. 5. The focus is equi-distant from the extremity of the sub-normal and the point of contact.

Because, $NR = 2FV$; therefore, $FR = 2FV + FN = FV + VN = TF = FM$.

Cor. 6. And hence, the focus is the centre of a circle which will pass through T, M, R.

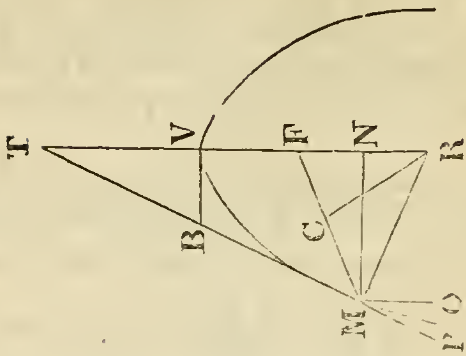
Cor. 7. If from the point of contact two lines be drawn, the one to the focus, and the other parallel to the axis, these lines will make equal angles with the tangent.

For, $FM = FT$ (Cor. 4), the angles F M T and F T M are equal; but the angles O M P and F T M are equal; therefore, $OMP = FMT$.

Cor. 8. A perpendicular from the end of the sub-normal to a line drawn from the point of contact to the focus, will cut off from that line a part equal to half the parameter.

Because the triangles R C F and N M F are similar, and have the equal sides F R and F M; therefore, $FC = NF$, and consequently, $MC = RN = \frac{1}{2}P$ (Cor. 3).

Cor. 9. A tangent at the vertex produced to meet any other tangent, is a mean proportional between half the parameter and half the abscissa.



For, by similar triangles V B is half of M N, V N being equal to T V. But $\frac{1}{4}$ M N^2 $= \frac{1}{2}$ V N $\times \frac{1}{2}$ *p*; therefore, V B^2 $= \frac{1}{2}$ V N $\times \frac{1}{2}$ *p*, (Prop. III.)

Cor. 10. Hence, the perpendicular V B is a mean proportional between the abscissa and $\frac{1}{4}$ the parameter.

V B^2 $= \frac{1}{2}$ V N $\times \frac{1}{2}$ P $= \frac{1}{4}$ P $\times$ V N.

V B^2 $= \frac{1}{2}$ V N $\times \frac{1}{2}$ P $= \frac{1}{4}$ P $\times$ V N.

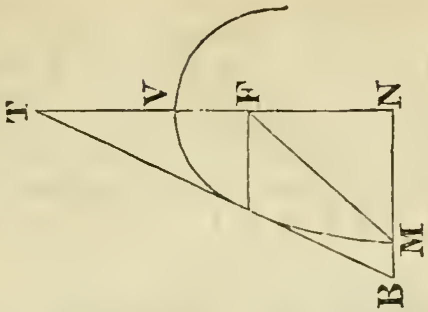
Cor. 11. Consequently, if the points F and B be joined, F B* will be at right angles to the tangent, on account of the similarity of the triangles T V B and T F B. Hence, it appears that a perpendicular from the focus to the tangent, and the tangent at the vertex will meet at the same point of the tangent, B.

Cor. 12. A perpendicular, from the focus to the tangent, is a mean proportional between the distance of the focus from the point of contact, and the focal distance.

Because, B F^2 $=$ F T $\times$ T V (similar triangles); but F T $=$ F M (Cor. 4.); therefore, F B^2 $=$ F M $\times$ F V.

Cor. 13. If any ordinate be produced to meet the focal tangent, the ordinate so produced will be equal to the distance of the focus from the point in which the ordinate intersects the curve.

* F B may be drawn with a pen.

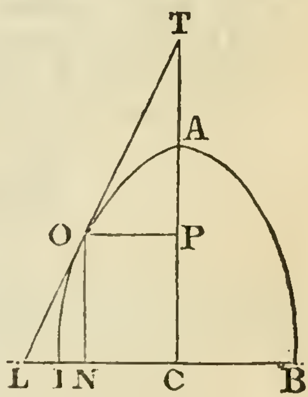


Because, M F = V N + V F = V N + V T = N T = N B (Cor. 4, and similar triangles).

PROPOSITION VIII.

If an ordinate be produced to meet the tangent, then, as the double ordinate passing through the point of contact is to the sum of the two ordinates, so is their difference to the difference added to the external part of the ordinate produced, that is, 2 P O : B N :: I N : N L.

Because, by similar triangles T P : P O :: O N : N L; but T P : P O :: 2 P O : P (Prop. VII. Cors. 1, 2); and B N : O N :: P : I N (Prop. IV.); therefore Ex. Equo. 2 P O : P :: O N : N L; and then, Ex. Equo. perturbate, 2 P O : B N :: I N : N L.



Cor. 1. The difference between the ordinates is a mean proportional between double the less and the external part of the lower ordinate. For, dividing the terms of the proportion, we get 2 P O : B N — 2 P O :: I N : N L — I N; that is, 2 P O : I N :: I N : L I.

Cor. 2. By this means, a tangent may be drawn from any given point L, in the ordinate produced. Because 2 O P =

$BI - 2NI$; therefore, $BI - 2NI : NI :: NI : LI$; hence, $NI^2 + 2NI \times IL = BI \times IL$; therefore, $NI^2 + 2NI \times IL + IL^2$, or $NL^2 = BL \times LI$; hence, $NL = \surd (BL \times LI)$ from which it appears that if the point L be given, the points N and O may be found.

Cor. 3. By composition, the Proposition becomes $2PO + IN : IN :: BN + NL : NL$; that is, $BN : IN :: BL : NL$.

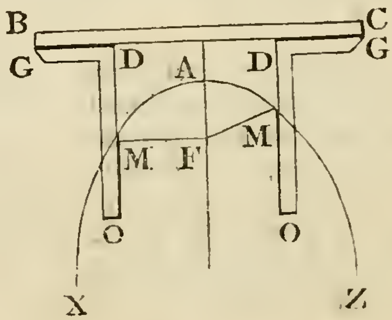
From the generation of the curve and Prop. II. it appears that two right lines drawn from any point in the curve, the one to the focus, and the other perpendicular to the directrix, will be always equal; hence, the following construction.

PROPOSITION IX.

PROBLEM.

To construct a parabola by motion.

Provide a ruler, such as B C, and fix it on the plane on which the parabola is to be described; to the directrix B C apply a square O D G, similar to what is commonly called a carpenter's square, in such a manner that one of its sides D G may lie close to B C; attach one end of a thread, F M O, equal in length to O D, to the end of the ruler at O, the other end of the thread O M F, being fixed at F;



then, slide the side of the square D G along the ruler B C, keeping the thread stretched by means of a pin M, with its

part M O close to the side of the square D O; and the curve A M X, described by the motion of the pin, will be one part of a parabola.

If the square be turned over, as represented in the figure, and moved on the other side of the fixed point F, the other part, A M Z, of the parabola will be described by the pin M.

Here O M + M F = O D, and taking away the common part O M, the remainders M D and M F will always remain equal, which is the property of the curve.

OF THE HYPERBOLA.

DEFINITIONS.

1. If in the line B A produced both ways, there be assumed two points, F and f, equi-distant from B A; and if, from the centres, F and f, with the radii B I and A I (I being assumed beyond A), arcs be described, so as to intersect each other in M, and an infinite number of such points be found, the curve passing through these points is called a hyperbola.

2. The points F and f are called the foci.

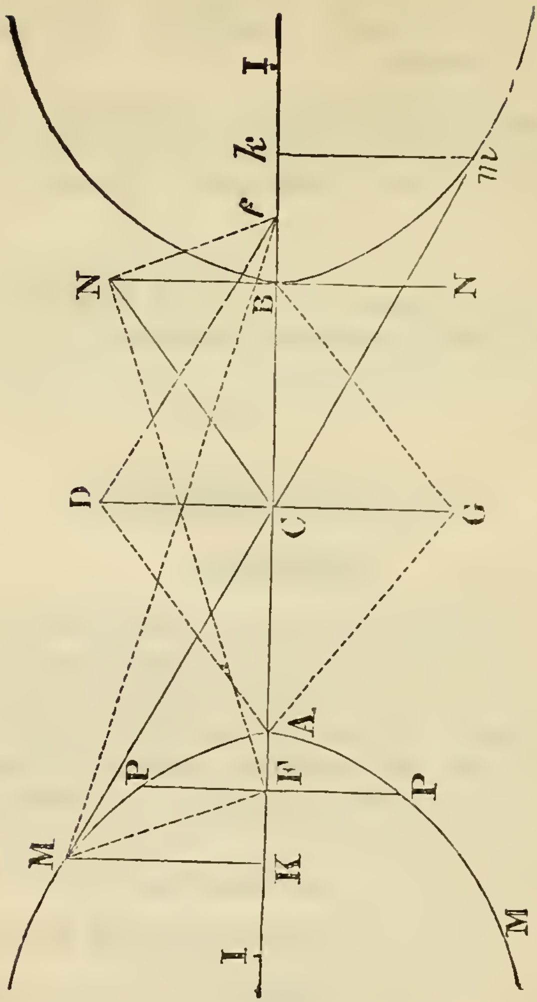
3. The line B A is called the transverse or greater axis.

4. The point C in the middle of B A is the centre.

5. The line D G, passing through the centre, perpendicular to the axis, and of such a length that A D and A G may be each equal to C F, is called the conjugate or less axis.

6. Any line passing through the centre, and terminated both ways by the curves, is called a diameter.

7. A right line drawn from the curve to a diameter pro-



duced and parallel to the tangent as its vertex, is called an ordinate.

8. The part of the axis, or of any diameter produced, which is intercepted between the vertex of that diameter and the crdinate, is called an abscissa.

9. The double ordinate P P, to the transverse axis, passing through the focus, is called the latus rectum.

PROPOSITION I.

The difference of right lines drawn from any point in the curve to the foci, is equal to the transverse axis, and therefore always equal.

For, f M $=$ B I, and
F M $=$ A I; therefore, f M — F M $=$ B I — A I $=$ B A

Cor. 1. Hence, it appears that the curve must pass through B and A.

For, F B — f B $=$ f A — F A $=$ B A.

Cor. 2. By the definition B A is bisected in C, and B f $=$ A F; therefore, the focal distance is bisected in C.

PROPOSITION II.

Half the conjugate axis is a mean proportional between the distance of the focus from the extremities of the transverse axis.

For, B G^2 — B C^2 $=$ (C G^2) $=$ f C^2 — B C^2 $=$ f B^2 $+$ B C^2 $+$ $2 f$ B $\times$ B C — B C^2 $=$ f B^2 $+$ $2 f$ B $\times$ B C $=$ (F B $+$ 2 B C) $\times$ f B $=$ F A $\times$ f B; that is, C G^2 $=$ f A $\times$ f B.—See the last figure.

PROPOSITION III.

If the conjugate axis be applied to the vertex of the transverse axis, the centre of the hyperbola will be the centre of a circle which will pass through the foci and the extremity of the conjugate.

For, by the construction, C N $=$ (B D) C F $=$ C f; therefore, if from the centre C, with the radius C F, a circle be

described, it will pass through N and f.—See the last figure.

Cor. Therefore, the distance of either focus from the extremity of the conjugate so applied, is a mean proportional between the distance of the foci, and the distance of that focus from the vertex to which the conjugate is applied. For, as a circle, with the centre C, passes through F, N, f, the angle $F N f$ is right; therefore, $f N^2 = F f \times f B$, and $F N^2 = F f \times F A$ (8. IV.).

PROPOSITION IV.

The latus rectum is a third proportional to the transverse and conjugate diameters.—See last figure.

For, $P F^2 + F f^2 = P f^2$*; but $F f = A B + 2 B F$, and $F f = A B + P F$. Then, $P f^2 = A B^2 + A B \times 2 P F + P F^2$, and $P f^2 = P F^2 + f F^2$; therefore, $A B \times 2 P F = f F^2 - A B^2 = 4 F C^2 - 4 B C^2 = 4 B D^2 - 4 B C^2 = 4 C D^2 = D G^2$; that is, $A B \times 2 P F = D G^2$, $\therefore A B : D G :: D G : (2 P F) P P$.

Cor. The distance between the foci is a mean proportional between the transverse, and the sum of the transverse and latus rectum.

For, $F f^2 = P f^2 - P F^2 = (A B + 2 P F) \times A B$; therefore, $A B : F f :: F f : (A B + 2 P F)$.

PROPOSITION V.

As the square of half the transverse is to the rectangle of the focal distances from the vertices, so is the rectangle of the abscissa and sum of the transverse and abscissa to the square of the ordinate; that is, $A C^2 : A F \times F B :: A K \times B K : m K^2$.

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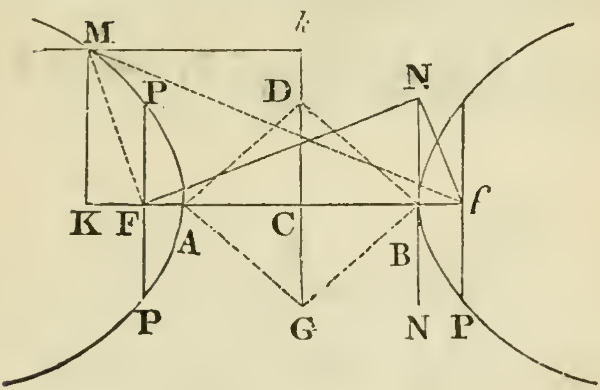
* $P f$, which is not in the figure, may be conceived to be drawn.

It is a well-known property, that the base of a triangle is to the sum of the sides, as the difference of the sides is to the sum or difference of the segments of the base made by a perpendicular on it, from the vertical angle, according as the perpendicular falls outside or inside the triangle. Alternating this proportion, and dividing by 2, we have $A\,C : F\,C :: C\,K : \frac{f\,M + F\,M}{2} = f\,M - A\,C$; which being squared and divided, will be $A\,C^2 : F\,C^2 - A\,C^2 :: C\,K^2 : f\,M^2 - 2\,f\,M \times A\,C + A\,C^2 - C\,K^2$; which, being inverted and divided, will be $A\,C^2 : F\,C^2 - A\,C^2 :: C\,K^2 - A\,C^2 : f\,M^2 - 2\,f\,M \times A\,C + 2\,A\,C^2 - C\,K^2 - F\,C^2$. But $F\,C^2 - A\,C^2 = A\,F \times F\,B$; $C\,K^2 - A\,C^2 = A\,K \times K\,B$; and $2\,f\,M \times A\,C - 2\,A\,C^2 = 2\,F\,C \times C\,K$ (by the first analogy), $\therefore f\,M^2 - 2\,f\,M \times A\,C + 2\,A\,C^2 - C\,K^2 - F\,C^2 = f\,M^2 - f\,K^2 = M\,K^2$; hence, $A\,C^2 : A\,F \times F\,B :: A\,K \times K\,B : M\,K^2$.



Cor. The square of half the transverse is to the square of half the conjugate, as the rectangle of the abscissa, and the sum of the abscissa and the transverse to the square of the ordinate.

For $C\,D^2 = A\,F \times F\,B$.

Cor. 2. $D\,G^2 = A\,B \times P$.

Cor. 3. Hence, the squares of the ordinates are proportional to the rectangles of the abscissa by the sum of the abscissa and transverse.

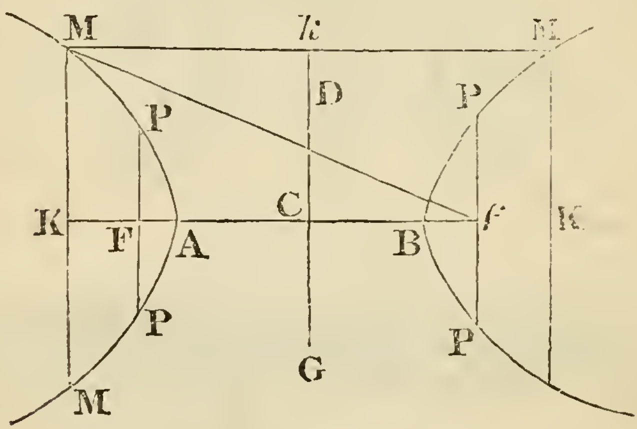
Cor. 4. The square of half the conjugate is to the square of half the transverse, as the sum of the squares of half the conjugate and the distance of the ordinate from the centre to the square of the ordinate of the conjugate.

By Cor. 1, $A\,C^2 : C\,D^2 :: C\,K^2 - A\,C^2 : M\,K^2$; hence, by inversion and composition, $C\,D^2 : A\,C^2 :: M\,K^2 + C\,D^2 : C\,K^2$, or $M\,k^2$.

Cor. 5. As the parameter or latus rectum is to the transverse, so is the sum of the squares of half the conjugate and the distance of the ordinate from the centre to the square of the ordinate of the conjugate.

Cor. 6. Therefore, these squares are as the squares of the ordinates of the conjugate.

Cor. 7. As $C\,D^2 = A\,F \times F\,B$; then, $A\,F \times F\,B : A\,C^2 :: C\,D^2 + M\,K^2 : M\,k^2$.



Cor. 8. And as $A\,C^2 = F\,C^2 - C\,D^2$; then, $A\,K \times K\,B : C\,F^2 - C\,D^2 :: M\,K^2 : C\,D^2$.

Cor. 9. Also, as $A B^2 = \frac{G D^4}{P^2}$; then $G D^2 : P^2 :: A K \times K B : M K^2 :: M k^2 : M K^2 + C D^2$.

PROPOSITION VI.

If through the extremities of any two ordinates, a right line be drawn so as to meet the axis; then as the square of the distance of one of the ordinates from the point of intersection is to the rectangle of the abscissa, by the abscissa and transverse, so is the sum of the distances of both ordinates from the point of intersection, to the sum of their distances from the centre; that is, $T Q^2 : A Q \times Q B :: T Q + T L : C Q + C L$.

For, by similar triangles, and Proposition V., $T Q^2 : A Q \times Q B :: T L^2 : A L \times L B$; therefore, by division, $T Q^2 : A Q \times Q B :: (T L^2 - T Q^2 : A L \times L B - A Q \times Q B ::) C L^2 - C Q^2 - 2 T C \times C L + 2 T C \times C Q : C L^2 - C Q^2$, and by dividing the two last terms by $C L - C Q$, we shall have $T Q^2 : A Q \times Q B :: (C L + C Q - 2 C T : C L + C Q ::) T L + T Q : C L + C Q$.

Cor. If P and M coincide, then, $A Q$ and $A L$ will become equal to $A N$, and the line $T M$ will be a tangent, as $T O$; therefore, $T N^2 : A N \times N B :: 2 T N : 2 C N$; hence, by dividing the two last by 2, and the first and third by $T N$, we shall get $C N : B N :: A N : T N$; which is the property of the tangent.

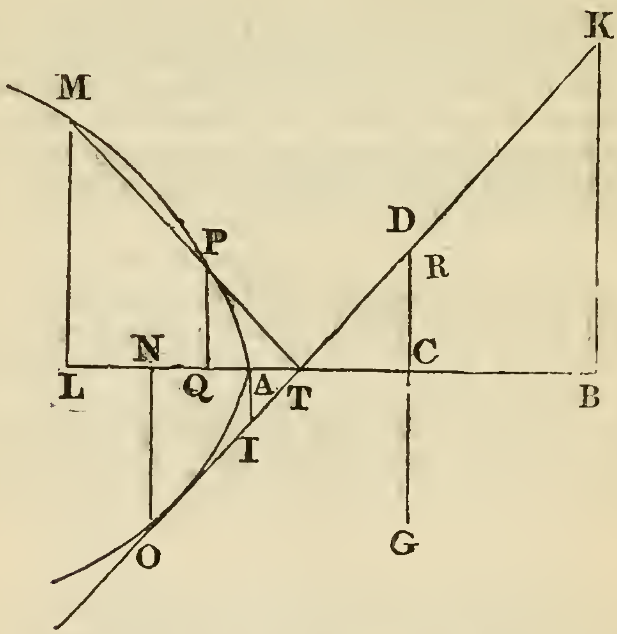
Cor. 2. By dividing the last analogy, we shall get $C N : B N - C N :: A N : T N - A N$; that is, $C N : A C :: A N : A T$.

Cor. 3. From the first, we get $C N : C N - A N :: B N : B N - T N$; that is, $C N : A C :: B N : B T$.

Cor. 4. Therefore, A N : A T :: B N : B T (Cors. 2, 3).

Cor. 5. From the third, we get C N : A C :: C N — A N : A C — A T; that is, C N : A C :: A C : C T.

Cor. 6. From the last analogy, we get C T · C A :: C T + C A : C A + C N; that is, C T : C A :: B T : B N.



Cor. 7. From the 5th Cor. we get C T : C A :: (A C — C T : C N — A C ::) A T : A N.

Cor. 8. From the 6th Cor. we get C T : B T :: A C — C T : B N — B T; that is, C T : B T :: A T : N T.

Cor. 9. By comparing the 6th and 8th Cors. we get A C : B N :: A T : N T.

Cor. 10. By comparing the 7th and 8th Cors. we get A C : A N :: B T : N T.

Cor. 11. If the point N be at an infinite distance, then A C will be equal to A T, and therefore, the tangent passes through the centre.

Cor. 12. If perpendiculars to the extremities of the transverse and semi-conjugate be produced to meet the tangent produced, then, A I : O N :: C R : B K (Cor. 8, and similar triangles).

Cor. 13. By similar triangles, T B : T N :: B K : N O; therefore, by Cor. 10, A C : A N :: B K : N O.

Cor. 14. By similar triangles, A T : T N :: I A : N O; therefore, by Cor. 9, A C : B N :: A I : N O.

Cor. 15. By Ex. Equo. perturbate, A N : B N :: A I : B K.

Cor. 16. By compounding the proportions in the 13th and 14th Cors., $A C^2 : A N \times N B :: A I \times B K : N O^2$ :: (Prop. V. Cor. 1) $C D^2 : N O^2$; therefore, $C D^2 = I A \times B K = A F \times F B$.

Cor. 17. When the tangent passes through the centre, then will A I and B K be equal; and therefore, C D = A I = B K (Cor. 16).

Cor. 18. Therefore, if half the conjugate be applied to the vertex of the transverse diameter, a tangent at an infinite distance will pass through the centre and the extremity of the conjugate.

Cor. 19. Therefore, if a right line be thus drawn, it will continually approach the curve, but will never meet it; being a tangent at an infinite distance.

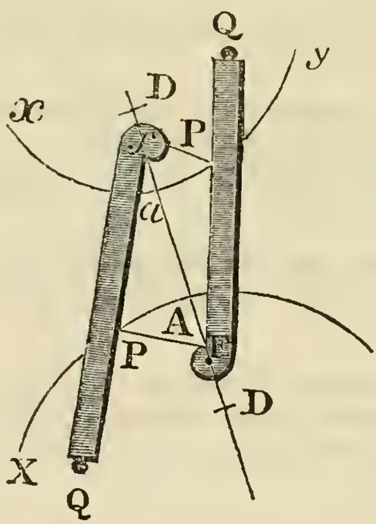
Besides the method given at the beginning of this chapter to construct a hyperbola by means of points, the following may be found useful in practice :—

Fasten one end of a long ruler *f* P Q at the point *f*, by means of a pin on a plane, so as to turn freely about *f* as a centre. Then fasten the end of a thread F P Q, shorter than the ruler, at the point F, and the other end of the thread at the end Q of the ruler.

Then, turn the ruler f P Q about the fixed point f, at the same time, keeping the thread tight, and its part P Q close to the side of the ruler, by means of the pin P; the curve line A x described by the pin P, is one part of an hyperbola.

If the ruler be turned, and moved on the other side of F, the other part A y of the same hyperbola will be described.

If one end of a thread be fixed to the end Q of the ruler, while the other end of the ruler is fixed at F, the hyperbola $x\ a\ y$ is described.



ARITHMETIC OF INFINITES.

To find the superficial or solid content of any figure, the pupil is requested to attend to the following preparatory Propositions, showing how to find the sum of certain progressional series.

PROPOSITION I.

In any series of equal numbers, as 1, 1, 1, *&c.,* 2, 2, 2, *&c.,* 3, 3, 3, *&c., the sum will be equal to one of the terms multiplied by the number of terms; that is,* $S = n\,a$, a, *being one of the terms, and* n, *the number of terms.*

PROPOSITION II.

In a series of numbers in arithmetical progression, beginning with a cypher, the common difference being 1, *the sum will be equal to half the product of the greater, and number of terms; that is, putting* $g =$ *the greatest term* $n =$ *the number of terms, and* $S =$ *the sum of the series:* $S = \frac{1}{2}\,n\,g$.

$0 + 1 + 2 + 3 + 4$; then, $S = 4 \times 5 \div 2 = 10$. For the reason of this, see *Arithmetic.*

PROPOSITION III.

In a series of squares, whose sides or roots form an arithmetical progression, differing by 1, *and commencing with a cypher; the sum of such a series is equal* $\frac{1}{3}$ *of the greatest term multiplied by the number of terms; when the series is infinitely continued; that is,* $S = \frac{1}{3}\,g^2\,n$.

1. Thus, $0 + 1 + 4 = 5$. But $4 \times 3 = 12$; then, $\frac{5}{12} = \frac{1}{3} + \frac{1}{12}$.
2. $0 + 1 + 4 + 9 = 14$. But $9 \times 4 = 36$; then, $\frac{14}{36} = \frac{1}{3} + \frac{1}{18}$.
3. $0 + 1 + 4 + 9 + 16 = 30$. But $16 \times 5 = 80$; then, $\frac{30}{80} = \frac{1}{3} + \frac{1}{24}$.

In the first series, where the number of terms is 3, the sum exceeds $\frac{1}{3}$ of the greatest term multiplied by the number

of terms, by $\frac{1}{12}$; in the second series, where the number of terms is four, the sum exceeds $\frac{1}{3}$ of the greatest term multiplied by the number of terms, by $\frac{1}{18}$; in the third series the excess of the sum above $\frac{1}{3}$ of the greatest term multiplied by the number of terms, is $\frac{1}{24}$; from which it appears, that the excess of the sum of the series above $\frac{1}{3}$ of the product of the greatest and number of terms, is continually diminishing, according as the number of terms increases; therefore, when the number of terms is infinite, the excess of the sum of the series above $\frac{1}{3}$ of the product of the greatest number of terms, must necessarily be infinitely small, and consequently less than any assignable quantity; which excess may then be considered as nothing; hence, the sum of the series, when the number of terms is infinite, is equal to $\frac{1}{3}$ of the product of the greatest term and number of terms.

PROPOSITION IV.

In a series of cubes, whose roots form an arithmetical progression, beginning with a cypher, the common difference being 1, *and the number of terms infinite, the sum will be equal to* $\frac{1}{4}$ *of the product of the greatest term, multiplied by the number of terms; that is,* $S = \frac{1}{4} g^3 n$.

1. Thus, $0 + 1 + 8 + 27 = 36$. But $27 \times 4 = 108$; then, $\frac{36}{108} = \frac{1}{4} + \frac{1}{12}$.

2. $0 + 1 + 8 + 27 + 64 = 100$. But $64 \times 5 = 320$; then, $\frac{100}{320} = \frac{1}{4} + \frac{1}{16}$.

3. $0 + 1 + 8 + 27 + 64 + 125 = 225$. But $125 \times 6 = 750$; then, $\frac{225}{750} = \frac{1}{4} + \frac{1}{20}$.

In the first series, the excess of the sum above $\frac{1}{4}$ of the greatest term and number of terms multiplied together, is $\frac{1}{12}$; in the second series, the excess is only $\frac{1}{16}$; and in the third, only $\frac{1}{20}$; therefore, it is obvious that when the number of terms is infinitely great, the excess must necessarily be in-

finitely small, and therefore less than any assignable quantity, which excess therefore may be considered as nothing. Hence, the truth of the proposition.

PROPOSITION V.

In a series of biquadrates, whose roots form an arithmetical progression, beginning with a cypher, the common difference being 1, *and the number of terms infinite, the sum of such a series will be equal to* $\frac{1}{5}$ *of the product of the greatest term and number of terms multiplied together; that is,* $S = \frac{1}{5} g^4 n$.

The truth of this Proposition may be proved, as in the foregoing Propositions, by showing that the excess of the sum of the series above the result of the greatest term and number of terms multiplied together, vanishes, when the number of terms becomes infinite.

PROPOSITION VI.

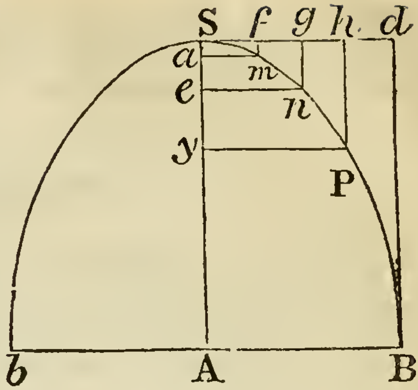
In any two ranks of proportionals, having the same number of terms, whether finite or infinite, the first term of one series is to the first term of the other, as the sum of all the terms of the first series to the sum of all the terms of the other. For the truth of this, see Arithmetic.

To apply the preceding Proposition to geometrical quantities, it will be necessary to suppose a line to consist, or to be composed of an infinite number of points: a surface of an infinite series of lines, either curved or straight: a solid of an infinite series of planes or superfices.

PROPOSITION VII.

The area of a parabola is equal to $\frac{2}{3}$ *of its circumscribed parallelogram.*

Draw B d parallel to A S, and S d parallel to A B, conceive S d to be divided into an infinite number of equal parts in the points f, g, h, &c., through which conceive a series of parallels to be drawn, such as $f\,m$, $g\,n$, h P, &c., meeting the semi-ordinates $a\,m$, $e\,n$, y P, &c., in the curve, at the points m, n, P, &c.



Then, from the property of the curve, (Prop. III. Cor. 1.) we have the following analogies, viz. :—

$$\mathrm{S\,A} : \mathrm{A\,B}^2 :: \mathrm{S}\,a : a\,m^2$$
$$\mathrm{S\,A} : \mathrm{A\,B}^2 :: \mathrm{S}\,e : e\,n^2$$
$$\mathrm{S\,A} : \mathrm{A\,B}^2 :: \mathrm{S}\,y : y\,p^2, \text{ \&c.}$$

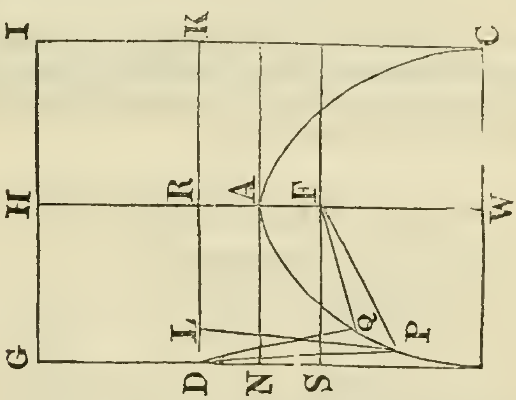
But S $a = f\,m$, S $e = g\,n$, S $y = h$ P, S A $= d$ B; therefore, by inversion, we have

$$\mathrm{A\,B}^2 : d\,\mathrm{B} :: y\,\mathrm{P}^2 : h\,\mathrm{P}$$
$$\mathrm{A\,B}^2 : d\,\mathrm{B} :: e\,n^2 : g\,n$$
$$\mathrm{A\,B}^2 : d\,\mathrm{B} :: a\,m^2 : f\,m, \text{ \&c.}$$

In these proportions, $a\,m^2$, $e\,n^2$, $y\,\mathrm{P}^2$, &c., are a series of squares, whose roots S f, S g, S h, &c., are in arithmetical progression, beginning with 0 at S, the common difference being 1, and number of terms infinite; and as the lines $f\,m$, $g\,n$, h P, &c., are as these squares having B d the greatest term, S d, the number of terms; then the sum of all the lines, by Prop. III. will be $\mathrm{S} = \dfrac{\mathrm{B}\,d \times \mathrm{S}\,d}{3}$; but S A × A B = B d × S d; therefore, $\dfrac{\mathrm{S\,A} \times \mathrm{A\,B}}{3} = (\mathrm{S})$ the sum of all the lines $f\,m$, $g\,n$, h P, &c., which constitute the space S d B S, outside the semi-parabola. But the area

of the parallelogram A S d B is S A $\times$ A B; therefore, S A $\times$ A B $- \frac{1}{3}$ S A $\times$ A B $= \frac{2}{3}$ S A $\times$ A B is the area of the semi-parabola A S P B; therefore the area of the whole parabola will be equal to $\frac{2}{3}$ S A $\times$ b B; but S A $\times$ b B is the area of the circumscribed parallelogram; hence, the area of the parabola is $\frac{2}{3}$ of its circumscribed parallelogram.

To prove that the parabola is $\frac{2}{3}$ of the circumscribed parallelogram by another method, take any point P in the curve infinitely near B (conceive B F to be drawn). Now, from the nature of the parabola, the angle P B D $=$ angle P B F, also B D $=$ B F, and P B common; therefore, (IV. 1,) the triangles P B F and P B D are equal. Again, L P Q and F P Q are equal; and consequently, their supplements are equal, viz., L P B $=$ F P B, and L P $=$ P F, by the property of the parabola, and P B common; therefore the triangles L P B and F P B are equal; hence, the



triangles D P P and L P B are equal; but the triangle L P B $=$ triangle D L P; therefore, the triangle D L P $=$ D P B; hence, the space D L P B is twice the triangle B L P, and therefore, the space D L P B $=$ twice the triangle B P F. In like manner it may be shown that the space D R A Q P B is twice the space B P Q A F B.

Make R H $=$ F W, then the parallelogram G R $=$ twice the triangle B F W; therefore, the space B A H G is double of A P B W A, and consequently, the parallelogram W G is three times the space W A B, that is, the space A B W

is $\frac{1}{3}$ of the parallelogram W G. But G R = W S and S A = A D $\therefore$ G W = 2 W N. Now, as the space A B W is $\frac{1}{3}$ G W, it is $\frac{2}{3}$ W N. Hence, the parabola B A C is $\frac{2}{3}$ of the parallelogram C N.

To prove what was assumed in the foregoing demonstration, viz., that the angle P B D is = the angle P B F. From the nature of the parabola P F = P L and the angle D P L being infinitely small, causes no sensible difference between P D and P L $\therefore$ P D = P F and B P is common in the two triangles; also B D = B F $\therefore$ (8. 1) the angle P B F = P B D.

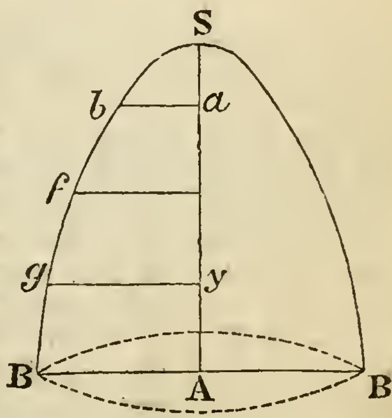
PROPOSITION VIII.

Every parabolic conoid is equal to half its circumscribed cylinder.

If the semi-parabola B S A be made to revolve about its axis S A, the solid thus formed is called a parabolic conoid, and may be conceived to be constituted of an infinite series of circles parallel to its base B B.

From the property of the curve, Prop. III. S A : A B :: A B : P $\left(= \frac{A\,B^2}{S\,A}\right)$ the parameter.

Then, $S\,a \times P = b\,a^2$
$S\,e \times P = f\,e^2$
$S\,y \times P = g\,y^2$



Here, if $S\,a$, $S\,e$, $S\,y$, &c., be a series in arithmetical progression, then, $S\,a \times P$, $S\,e \times P$, $S\,y \times P$, &c., are in arithmetical progression, therefore, $b\,a^2, f\,e^2, g\,y^2$, are a series in arithmetical progression, beginning at S, the first term being 0, the common difference 1, A B^2 the greatest erm, and S A the number of erms. Therefore, A $B^2 \times \frac{1}{2}$

S A = S, the sum of the series (Prop. II.). But the areas of the circles, which constitute the solid, and whose radii are $b\,a, f\,e, g\,y$, &c., are, $(2\,b\,a)^2 \times n, (2\,f\,e)^2 \times n, (2\,g\,y)^2 \times n$, n being equal to ·7854; therefore, putting $d = 2$ A B, and $h =$ S A, the sum of all the circular areas constituting the parabolic conoid will be $\frac{1}{2}\,n\,d^2\,h$. But $n\,d^2\,h$ is the content of the cylinder, the diameter of whose base is d, and height h; therefore, the parabolic conoid is half of its circumscribed cylinder.

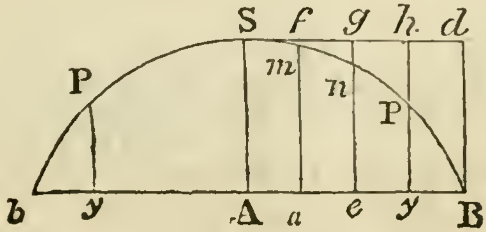
Cor. The solidity of the lower frustum of a conoid cut off by a plane parallel to the base, is equal to half the sum of the areas of both bases, multiplied by the height of the frustum.

It has been shown in the Proposition, that the areas of the circles which constitute the frustum are a series in arithmetical progression, the sum of which is equal to $\frac{1}{2}$ the sum of the extremes multiplied by the number of the terms; but the extremes are the areas of the two bases of the frustum and the height the number of terms; therefore, $S = \frac{1}{2}\,(A + a) \times h$, S being the solidity, A the area of the greater base, a the area of the less base, and h the height of the frustum.

PROPOSITION IX.

Every parabolic spindle is equal to $\frac{8}{15}$ of its circumscribed cylinder.

Draw S d parallel to A B, and also, $f\ a$, $g\ e$, $h\ y$, &c., parallel to A S. It has been proved in Proposition VII. that the lines $f\,m$, $g\ n$, h P, &c., are as a series of squares, whose roots



form an arithmetical progression; therefore, the squares, viz., $f\,m^2$, $g\ n^2$, $h\ P^2$, will be as a series of biquadrates, whose roots will form an arithmetical progression.

Now, the spindle is generated by the segment b S B revolving about b B; and the solid itself is composed of the series of circles whose radii are $m\ a$, $n\ e$, P y, &c.

Again, $S\,A - f\,m = m\,a$
$S\,A - g\,n = n\,e$
$S\,A - h\,P = P\,y$, &c.; therefore, by squaring,

1. $S\,A^2 - 2\,S\,A \times f\,m + f\,m^2 = m\,a^2$
2. $S\,A^2 - 2\,S\,A \times g\,n + g\,n^2 = n\,e^2$
3. $S\,A^2 - 2\,S\,A \times h\,P + h\,P^2 = P\,y^2$, &c.

In these equations, S A^2, S A^2, S A^2, &c., form a series of equal squares, of which A B is the number of terms; therefore, their sum will be $S\,A^2 \times A\,B$.

And because $f\,m$, $g\,n$, h P, &c., are a series of squares, of which S A is the greatest term, and A B the number of terms; their sum will be $\frac{S\,A \times A\,B}{3}$, (Prop. III.) which being multiplied by 2 S A, will give the sum of that part of the equation, $2\,S\,A \times f\,m$, $2\,S\,A \times g\,n$, $2\,S\,A \times h\,P$, &c., viz., $\frac{2\,S\,A^2 \times A\,B}{3}$.

Again, $f\ m^2$, $g\ n^2$, $h\ P^2$, &c., are a series of terms of biquadrates, as has been shown above, whereof $d\ B^2$, or S A^2 is the greatest, and A B the number of terms; therefore, their sum (by Prop. V.) will be $\frac{S\,A^2 \times A\,B}{5}$. Hence, it is obvious, that the sum of $m\ a^2$, $n\ e^2$, P y^2, &c, will be

$$S A^2 \times A B - \frac{2 S A^2 \times 2 A B}{3} + \frac{S A^2 \times A B}{5} = \frac{6 S A^2 \times A B}{5} - \frac{2 S A^2 \times A B}{3} = \frac{8 S A^2 \times A B}{15}.$$

But the areas of the circles, whose radii are S A, *m a*, *n e*, P *y*, &c., are found by multiplying the squares of their diameters by ·7854 ($= n$); therefore, the sum of double such a series of circles is $\frac{8 n D^2 H}{15}$, putting $D = 2 S A$, $H = 2 A B =$ the solidity of the whole spindle.

But the solidity of the circumscribed cylinder is $n D^2 H$; therefore, the solidity of the spindle, viz., $\frac{8 n D^2 H}{15}$ is the eight-fifteenth of its circumscribing cylinder.

Cor. From this may be derived a method of finding the solidity of the frustum, S A *y p*, of a spindle.

The area of a circle, whose radius is S A, being the greatest term, and the area of the circle whose radius is P *y*, the least term, and A *y* the number of terms; then the sum of such a series, that is, the sum of all the circles included between A and *y*, will be the solidity of the required frustum.

From what has been shown in the Proposition, the sum of all the series $S A^2$, $m a^2$, $g n^2$, $P y^2$, is

$$\left(S A^2 - \frac{2 S A \times h p}{3} + \frac{h p^2}{5}\right) \times A y \text{ which let } = Z.$$

By multiplying the equation by 3, we get

$$\left(3 S A - 2 S A \times h p + \frac{3 h p}{5}\right) \times A y = 3 Z.$$

Divide both sides of the equation by A y, and $3 \text{ S A}^2 - 2 \text{ S A} \times h\,p + \frac{3\,h\,p^2}{5} = \frac{3\text{ Z}}{\text{A}\,y}$; but $\text{S A}^2 - 2 \text{ S A} \times h\,p = (\text{S A} - 2\,h\,p)\text{ S A} = (p\,y - h\,p) \times (p\,y + h\,p) = p\,y^2 - h\,p^2$; then the difference of these will be equal; that is, $2 \text{ S A}^2 + \frac{3\,h\,p^2}{5} = \frac{3\text{ Z}}{\text{A}\,y} - p\,y^2 + h\,p^2$, and by transposition, $2 \text{ S A}^2 + p\,y^2 - \frac{2}{5}\,h\,p^2 = \frac{3\text{ Z}}{\text{A}\,y}$; divide by $\frac{3}{\text{A}\,y}$, and we get $(2 \text{ S A}^2 + p\,y^2 - \frac{2}{5}\,h\,p^2)\,\frac{1}{3}\,\text{A}\,y = \text{Z}$. Then, putting $\text{D} = 2 \text{ S A}$, $\text{C} = 2\,p\,y$, $d = 2\,h\,p$, and $\text{L} = \text{A}\,y$; $(2\,\text{D}^2 + \text{C}^2 - \frac{4}{10}\,d^2) \times \text{L} \times \cdot 2618$ is the solidity of the frustum.

PROPOSITION X.

A solid, formed by the rotation of an hyperbola upon its axis, is to the circumscribed cylinder, as half the transverse + $\frac{1}{3}$ the abscissa, to the sum of the transverse and abscissa.

Let t and x be the transverse and abscissa, P the parameter, n a quantity, by which, if the square of any radius be multiplied, the product will be the area of the circle.—See Fig. Prop. V.

By Prop. V. $m\,\text{K}^2 = \frac{\text{P}}{t} \times (t\,x + x^2)$; therefore, the area of a circle whose radius is m K, will be $\frac{n\,\text{P}}{t} \times (t\,x + x^2)$, and the sum of all the circles between A and K must be $\frac{n\,\text{P}\,x^2}{t} \times (\frac{1}{2}\,t + \frac{1}{3}\,x)$, but the solidity of the circumscribing

cylinder $\frac{n \text{ P } x^2}{t} \times (t + x)$; therefore, the hyperbolic conoid is to the cylinder as $\frac{1}{2} t + \frac{1}{3} x$ to $t + x$.

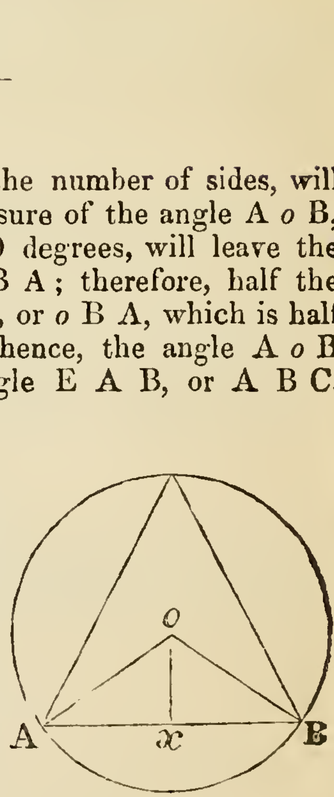
Cor. 1. $3 t + 2 x$ will be to $6 t + 6 x$ in the same ratio.

Cor. 2. When the abscissa becomes equal to the transverse, the conoid will be to the cylinder as 5 to 12.

DEMONSTRATIONS.

Dem. 1. 360 degrees divided by the number of sides, will give the arc A B, which is the measure of the angle A o B, and A o B being deducted from 180 degrees, will leave the sum of the equal angles o A B, o B A; therefore, half the remainder will give the angle o A B, or o B A, which is half of the angle E A B or A B C; hence, the angle A o B taken from 180, will leave the angle E A B, or A B C. (Page 18.)*

2. Having the side A B, which in each of the polygons is 1, we can easily discover the side o B, the radius of the circumscribed circle: thus, let fall the perpendicular $o\ x$, which bisects A B, (3 III. Euc.) then say, as sine of the angle $x\ o$ B, which is $\frac{1}{2}$ A o B, is to $\cdot 5$, ($= \frac{1}{2}$ A B $= x$ B) so is radius to o B. (See Trigonometry.) In the trigon, the angle $x\ o$ B is 60^{0}; then,



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* This and similar references are to the pages of the Mensuration wqere the rules are given, and where also the figures, unless given with the Demonstrations, are to be found.

As sine x o B 60°, ...	9·937531
is to x B ·5, ...	1·698970
so is radius 90°, ...	10·000000
	9·698970
	9·937531
to o B ·5773503, ...	—1·761439.

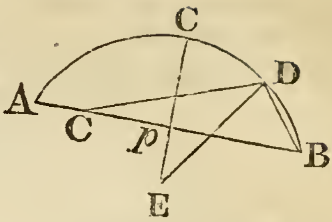
Now, to find the radius of the circumscribed circle, that is o B, when the side of the triangle is of any given length, as 10 yards, 10 miles, &c. From the property of similar triangles, 1 : ·5773503 :: 10 (= A B) : 10 × ·5773503 (= o B). The rest of the tabular numbers may be found in a similar manner. The reason of multiplying the side of any polygon by the number corresponding to it in the third column of the table, may be seen from the last analogy. (Page 18.)

3. Let the circumference be denoted by C, and let n denote the number of sides in the polygon; also let A B be divided into n parts; join C o.

The angle A o C $\left(= \frac{C}{n}\right)$ is given, as also the angle o A C, or o C A $\left(= 90 — \frac{C}{2\,n}\right)$ is given; A o $\left(= \frac{n}{2}\right)$ being given, o z $\left(= \frac{n}{2} — 2\right)$ is given; therefore C z can be found, as also the angle A z C, or its equal o z x; hence, the compliment of the angle, o z x, viz., o x z can be found; and o z being given, o x can be found; and hence x y can be found, which will be found to be $\frac{3}{4}$ of o y nearly; hence, the construction is obvious. (Page 19.)

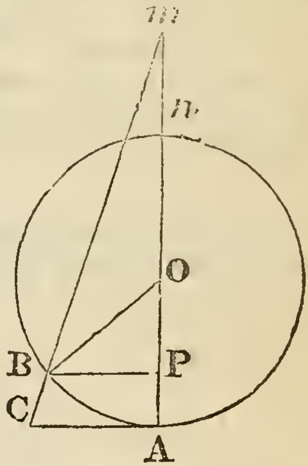
4. Draw E G at right angles to A B, E being the centre, join E B and E D. Let the radius E B and G p height of the arc be given; then E p is given; and as E B, E D, and D B are given, the angle E B D can be found; and as E B,

B p, and E p are given, the angle E B p can be found; and hence the angle C B can be found. Now, as C B, B D, and the angle C B D are given, the side C D can be found, which will be found to be equal to half the arc nearly. (Page 20.)



5. Let the line A C be assumed equal to the arc A B, join C and the extremity B, of the given arc, and produce C B to meet the production of A o at m.

Now, because the arc is given, its sine B P is given, and also A C; then, by similar triangles, A C : B P :: A m : P m; and by division, A C — P B : B P :: A m — P m (= A P) : P m; and A C — B P, P B, and A P (this being the versed sine of the arc), are given; hence, P n, and therefore m n can be found, which, when compared with the radius o n, will be $\frac{3}{4}$ of it nearly. Hence, the reason of the construction. (Page 20.)



6. $AB \times AS = AC^2$ (8. VI.); and the diameter of a circle being to its circumference as 7 to 22, nearly, as will be shown hereafter; then $7 : 22 :: \frac{AB}{7} : \times 22 =$ the circumference of the circle, whose diameter is A B. It will be shown hereafter, that half the diameter multiplied by half the circumference will give the area of the circle; therefore, $\frac{AB}{7} \times 11 \times \frac{AB}{2} = \frac{14}{7} \times 11 \times \frac{14}{2} = 2 \times 11 \times \frac{14}{2} = 14 \times 11 = AB \times AS = AC^2$. Hence, it

appears that the square of A C, viz., A C D G, is nearly equal to the area of a circle, whose diameter A B contains 14 parts, and the distance A S, from the perpendicular, 11 of such parts. And if 7 and 22 were strictly to each other as the diameter of a circle to its circumference, the square of A C would be the true area of the circle whose diameter is A B; but the ratio of 7 to 22 expresses only the approximate ratio of the diameter to the circumference, these being incommensurable; therefore, the square of A C only approaches the area of the circle, differing, however, from the truth only by a very small quantity. (Page 20.)

Fig. 2.

7. C F being divided into ten equal parts; it follows that if C F be a unit, F Z, Z 2, &c., will be each 1-10th of a unit; if C F be 10, F Z, Z 2, &c., will be each 1. Again, let the triangle E F x (Fig. 2, above), represent the triangle E F x (Fig. 1, page 21, Mensuration), and let E x be 1, then, by similar triangles, F E (10) : E x (1) :: F 1 (1) : 1 a; that is, 10 : 1 :: 1 : $\frac{1}{10}$ = 1 a. Also, F E (10) : E x (1) :: F 2 (2) : 2 b; that is, 10 : 1 :: 2 : $\frac{2}{10}$ = 2 b. Hence, it appears that the three divisions form a continued proportion, the ratio being 10. (Page 21.)

8. If the length of a rectangle be 6 feet, and the breadth 1 foot, it is evident that the rectangle will contain 6 *square* feet; if the breadth be 2 feet, it will contain 12 *square* feet; and so on. Hence, it is evident that the rectangle will contain 6 square feet as often as, or the *number of times* that, the breadth measures 1 foot; and generally if the length of a rectangle be a feet, and breadth b feet, the area will be a square feet, taken as often as b contains 1 foot. If a and b, or either of them, contain fractions of a foot, these fractions are, in Cross Multiplication, expressed *duodecimally;* that is, by a system of numeration, in which 12 is the base instead of 10; and the same rule may be observed as in the multiplication of decimals, namely, there must be as many duodecimals in the product as in both the multiplier and mul-

tiplicand. To begin to multiply by the highest denomination of the multiplier, as is usual, is not necessary; we might proceed from right to left with both multiplier and multiplicand, skipping a place, as in decimals. What are usually denominated *inches*, in the result, are not inches in reality, they are twelfths of square feet, whereas a square inch is $\frac{1}{144}$ of a square foot. A similar remark will apply to what are usually called parts, &c. (Page 25.)

9. Because the parallelograms A B C D, and A B F E, are equal (35, I.); but the area of A B F E is found (Problem II), by multiplying its length by its breadth; that is, the area of A B F E is equal to A B $\times$ B F $=$ D C $\times$ A E, which is the rule.

Note.—The continual product of any two sides of a parallelogram, and the natural sine of their contained angle will give the area; that is, A D $\times$ D C $\times$ by the natural sine of the angle D, will give the area. Because, A D : A E : : 1 : the natural sine of the angle D $\therefore$ A E $=$ A D $\times$ by the natural sine of D $\therefore$ A C $\times$ D C; that is, the area $=$ A D $\times$ D C $\times$ natural sine of D. (Page 29.)

10. The product of the base by the perpendicular height gives the area of a rectangular parallelogram, whose sides are equal to the base and perpendicular of the triangle (Prob. II.); but the triangle being half of the parallelogram (41, I.), it follows that half the product of the base and perpendicular will give its area. (Page 29.)

11. Put A C $= a$, A B $= b$, B C $= c$, and let half the sum of the sides be denoted by S; that is, S $= \frac{\text{A B} + \text{B C} + \text{A C}}{2}$, and let T equal the area of the triangle; then, 2 A C $\times$ C D $= a^2 + c^2 - b^2$ (13, II.); therefore, C D $= \frac{a^2 + c^2 - b^2}{2\,a}$. Again, B D$^2 =$ B C^2 $-$ C D^2 (47, I.) $= c^2 - \left(\frac{a^2 + c^2 - b^2}{2\,a}\right)$; but (Prob.

IV.) $T = \frac{A\,C \times B\,D}{2}$; therefore, $T^2 = \frac{A\,C^2 \times B\,D^2}{4} = \frac{4\,a^2\,c^2 - (a^2 + c^2 - b^2)^2}{16}$; but this expression, consisting of the difference of two squares, $\left(\frac{2\,a\,c}{4}\right)^2$ and $\left(\frac{a^2 + c^2 - b^2}{4}\right)^2$: which being equal to the rectangle under their sum and difference (Cor. 5, II.), may be transformed into the following expression, viz., $T^2 = \frac{2ac + a^2 + c^2 - b^2}{4} \times \frac{2ac - a^2 - c^2 \times b^2}{4}$ $= \frac{(a + c)^2 - b^2}{4} \times \frac{b^2 - (a - c)^2}{4}$; and by decomposing these factors again, we get $T^2 = \frac{a + b + c}{2} \times \frac{a - b + c}{2}$ $\times \frac{a + b - c}{2} \times \frac{-a + b + c}{2}$. From the assumption, $S = \frac{a + b + c}{2}$, $S - b = \frac{a - b + c}{2}$, $S - c = \frac{a + b - c}{2}$, and $S - c = \frac{-a + b + c}{2}$. Hence, by substitution, $T^2 = S \times (S - a) \times (S - b) \times (S - c)$: and $T = \surd\,(S \times (S - a) \times (S - b) \times (S - c))$, which is the rule. (Page 30.)

If two sides of any triangle, and the contained angle be given, the area may be found by multiplying the sides together, and the product by the natural sine of the contained angle, and dividing the result by 2. This is evident from Demonstration 10, and note to Demonstration 9. (Page 239.)

12. This rule is a corollary to the 5, II. Elrington's Euclid. (Page 30.)

13. $C\,B^2 - B\,D^2 = C\,D^2$ (47, I.); but $B\,D^2 = \frac{C\,B^2}{4}$

(Cors. 26. I. and 4. II.) therefore, $\left(CB^2 - \frac{CB^2}{4}\right) \times \frac{CB^2}{4}$ = the square of the area as may be derived from Prob. IV. Hence, $\sqrt{\left(\left(CB^2 - \frac{CB^2}{4}\right) \times \frac{CB^2}{4}\right)}$ = area of the triangle, which is the rule. (Page 32.)

14. By the foregoing Demonstration, the area of an equilateral triangle, each of whose sides measures 1, is $\frac{\sqrt{3}}{4}$; and similar triangles being to each other as the squares of their homologous sides (19. VI.), we have $1^2 : AB^2 :: \frac{\sqrt{3}}{4} : \frac{AB^2}{4} \times \sqrt{3}$. (Page 32.)

15. $CD \times \frac{AB}{2}$ = area (Prob. IV.) $\therefore \frac{AB}{2} = \frac{\text{area}}{CD}$, and $AB = \frac{\text{area}}{CD} \times 2$. (Page 32.)

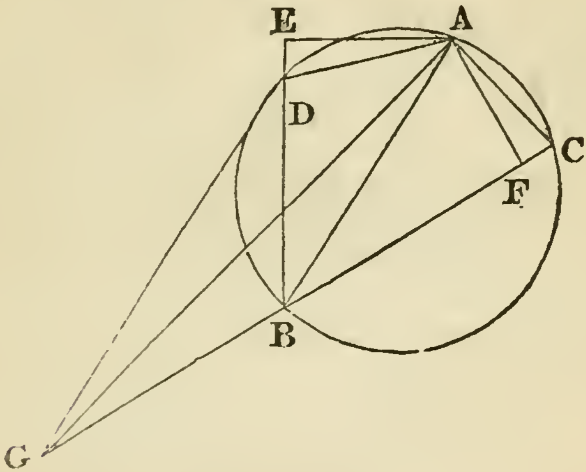
16. The truth of this rule is evident from 47. I., and from a Cor. to 5. II., which says, that the rectangle under the sum and difference of two quantities is equal to the difference of their squares. (Page 33.)

17. By the 8. VI. $AB \times BD = BC^2$. Hence, $BD = \frac{BC^2}{AB}$, which is one part of the rule. Again, $AD \times DB = DC^2$ (8. VI.); hence, $DC = \sqrt{AD = DB}$, which is the second part. (Page 34.)

18. By Prob. IV. the area of the triangle $ABC = \frac{AC \times BF}{2}$, and that of the triangle $ADC = \frac{AC \times DE}{2}$ Then the sum of these two areas will be the area of the

trapezium, viz., $\frac{A\,C \times B\,F}{2} + \frac{A\,C \times D\,E}{2} = \frac{B\,F + D\,E}{2}$ $\times$ A C, which is the rule. (Page 35.)

12. Let A C $= a$, C B $= b$, B D $= c$, and A D $= d$;

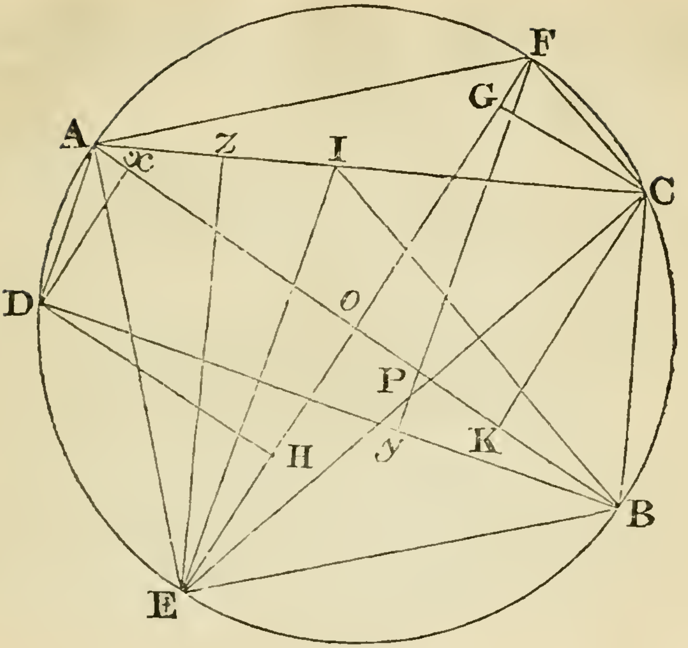


also, let the diagonal A B $= x$. Draw D G parallel to A B meeting C B produced in G; join G A; and from A draw the perpendicular A F (p), A E (p'). Because D G is parallel to A B, the triangles A B D and A B G are equal (37. I.); to each add the triangle A B C, then the triangle A G C is equal to the quadrilateral A D B C, inscribed in the circle. Now, as the triangles A G B, A D B, are equal, their bases G B, B D, are reciprocally proportional to their perpendiculars p and p'; that is, D B : B G :: p : p'. But the triangles, A D E, A F C are similar, being right-angled at E and F, and the angles A D E, and A C F, supplements of the angle A D B; therefore, (4. VI.) $p : p' :: a : d \therefore c$: B G :: $a : d$; hence, G B $= \frac{c\,d}{a}$, and $x^2 = a^2 + b^2 -$ $2\,b \times$ C F (13. II.); also, $x^2 = c^2 + d^2 + 2\,c$ $+$ D E (12. II.); therefore, $2\,b \times$ C F $= a^2 + b^2 -$ $(c^2 + d^2) - 2\,c \times$ D E; but $a : d ::$ C F : D E $\therefore$ D E $= \frac{d \times C\,F}{a}$; then, by substitution and transposition, we

get $CF = \frac{a}{2(ab+cd)} \times (a^2 + b^2 - c^2 - d^2)$; but $p = \sqrt{(a^2 - CF^2)} = \left\{ a^2 - a^2 \left(\frac{a^2+b^2-c^2-d^2}{2(ab+cd)} \right)^2 \right\}^{\frac{1}{2}}$ or $p = \frac{a}{2(ab+cd)} \left\{ (2ab+2cd)^2 - (a^2+b^2 - c^2 - d^2)^2 \right\}^{\frac{1}{2}}$; but $\frac{p \times CG}{2}$, or $\frac{p}{2}\left(b + \frac{cd}{a}\right)$ or $\left(\frac{ab+cd}{2a}\right) \times p =$ the area $=$ (by substitution) $\frac{1}{4}\left\{ (2ab+cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 \right\}^{\frac{1}{2}} = \frac{1}{4}\left\{ (2ab + 2cd + a^2 + b^2 - c^2 - d^2) \times (2ab + 2cd - a^2 - b^2 + c^2 + d^2) \right\}^{\frac{1}{2}} = \frac{1}{4}\left\{ (a^2 + 2ab + b^2 - c^2 + 2cd - d^2) \times (-a^2 + 2ab - b^2 + c^2 + 2cd + d^2) \right\}^{\frac{1}{2}} = \frac{1}{4}\left\{ [(a+b^2) - (c-d)^2] \times [-(a-b)^2 + (c+d)^2] \right\}^{\frac{1}{2}} = \frac{1}{4}\left\{ (a + b + c - d) \times (a + b - c + d) \times (a - b + c + d) \times (-a + b + c + d) \right\}^{\frac{1}{2}} = \left\{ \left(\frac{a+b+c+d}{2} - d\right) \times \left(\frac{a+b+c+d}{2} - c\right) \times \left(\frac{a+b+c+d}{2} - b\right) \times \left(\frac{a+b+c+d}{2} - a\right) \right\}^{\frac{1}{2}} =$ (putting $S = a + b + c + d$) $\left\{ \frac{S}{2} - d \right) \times \left(\frac{S}{2} - c\right) \times \left(\frac{S}{2} - b\right) \times \left(\frac{S}{2} - a\right) \right\}^{\frac{1}{2}}$ which is the rule. (Page 36.)

This may be demonstrated geometrically, thus:—

Let D A C B be the inscribed trapezium; bisect the angle A C B by C E, and from the point E demit the perpendicular E o F, on the line A B, which will evidently



bisect it, and also be the diameter of the circle; join E A, E B, and A F; from C let fall the perpendicular C G, and from E, the perpendicular E z, on the line A C. Make C I = C B; join B I, I E. Then, I E, and B E, the bases of the triangles, I C E, B C E, are equal; (4, I.) $\therefore$ I E = A E; and, therefore, A z = $\frac{1}{2}$ A I; that is, A z = half the difference between A C and C B; $\therefore$ C z = half their sum. For the same reason, D y is half the sum of the lines A D and D B, and B y half their difference; therefore, C z + D y = half the sum of the four lines, A C, C B, B D, D A; that is, C z + D y = half the perimeter of the inscribed trapezium A C B D. Then, if from C z + D y, each side of the trapezium be subtracted, respectively, the four remainders will evidently be D y — A z, D y + A z, C z — B y, C z + B y; which, being multiplied together, will give (D y^2 — A z^2 $\times$ (C z^2 — B y^2).

But A z^2 = F G $\times$ o E. For the triangles A z E and F G C being similar, A z^2 : z E^2 :: F G^2 : G C^2 :: F G^2

$: FG \times GE :: FG : GE$. And $Az^2 + zE^2 (= AE^2) : Az^2 :: FG + GE (= FE) : FE) : FG :: FE \times oE : FG \times oE$, but $FE \times oE = AE^2$, FAE being a right angle; therefore, $FG \times oE = Az^2$.

Again, $Cz^2 = Fo \times GE$. For it is obvious that $Cz^2 = CE^2 - EA^2 + Az^2 = GE \times FE - oE \times FE + oE \times FG = GE \times FE - GE \times oE = GE \times (FE - oE) = GE \times Fo$.

In a similar manner it may be proved that $By^2 = Fo \times EH$, and $Dy^2 = FH \times Eo$; therefore, $(Dy^2 - Az^2) \times (Cz^2 - By^2)$, the continual product of the four remainders, $= (FH \times Eo - FG \times Eo) \times (GE \times Fo - EH \times Fo) = ((FH - FG) \times Eo) \times ((GE - EH) \times Fo) = GH \times Eo \times GH \times Fo = GH^2 \times Eo \times Fo = GH^2 \times oB^2 =$ square of the area of the trapezium; because $CK \times oB =$ area of the triangle ABC, and $Dx \times oB =$ area of the triangle ADB, then $(CK + Dx) \times oB =$ area of the trapezium, but $CK + Dx = HG$.

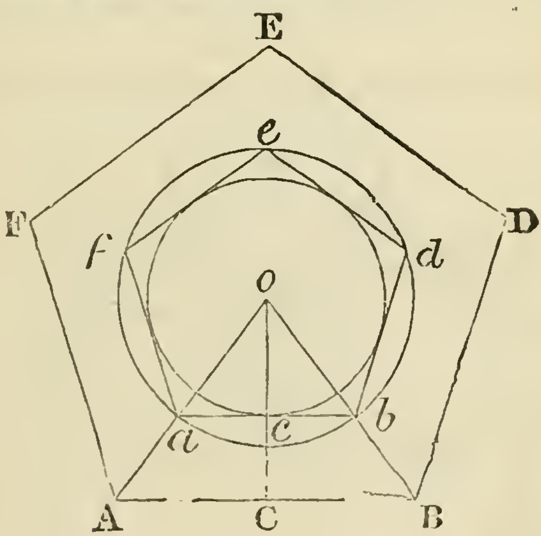
2). Bisect A D in G: through G draw E F parallel to C B, and G H parallel to A B; produce C D to E. The triangles A G F and E G D are evidently equi-angular, and the sides A G and G D equal; therefore, these triangles are equal (26. I.). Hence, the trapezoid A B C D is equal to the parallelogram, F E C B, but the area of the parallelogram is equal $FB \times CP$ (Prob. III.). Again, it has been shown that the triangles E D G, and A G F are equal in every respect; hence, $ED = AF$, and $EC = FB = GH$ (34. I.); therefore, G H is half the sum of E C and F B; that is $GH = \frac{EC + FB}{2} = \frac{DC + AB}{2}$; wherefore, $\frac{DC + AB}{2} \times CP$ gives the area of the trapezoid. (Page 37.)

2'. Rule I. Every regular polygon may be divided into as many equal triangles as the figure has sides, by joining its centre to the vertices of its angles; therefore the area of one

of the triangles multiplied by the number of sides, will give the area of the polygon. But the area of one of the triangles is found by multiplying the perpendicular by half the base (Prob. IV.); therefore, the area of the whole polygon is equal to the product of the perpendicular and half the sum of the sides.

Rule II. Regular polygons having the same number of sides, are similar to each other; and similar figures being to each other as the square of their like sides (20. VI.); therefore, as the areas in the Table are those of polygons whose sides are 1, it follows that 1^2 is to the area in the Table as the square of the side of the polygon to its area.

Rule III. The perpendiculars from the centres of two regular, similar polygons, will evidently be proportional to their sides; therefore, 1 : perpendicular in the Table :: the side of the polygon : its perpendicular.



The polygons A B D E F, and *a b d e f* are similar in every respect, and the area and perpendicular of the small polygon are found in the Table to the side 1.

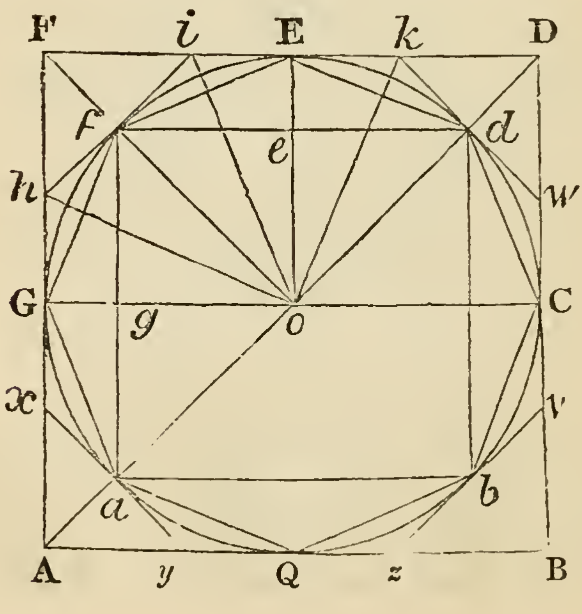
The tabular numbers are found by Trigonometry,—thus

for the pentagon: divide 360 degrees by 5, and the quotient is 72 degrees for the angle *a o b*; its half, 36 degrees, is the angle *a o c*. Then,

Sine angle $a\ o\ c = 36^{\circ}$ . . .	9·7692187
is to $a\ c = \cdot 5\ (= \frac{a\ b}{2} = \frac{1}{2})$ . .	–1·6989700
so is sine angle $o\ a\ c = 54^{\circ}$ . .	9·9079576
	9·6069276
	9·7692187
to perpendicular $c\ o = \cdot 688191$ ·	–1·8377089

Hence, $(\cdot 688191 \times 5) \div 2 = 1\cdot 720477$ = the area of the pentagon whose side is 1. The rest of the numbers in the Table are calculated in a similar manner. These may be found by other methods, some of which are rather difficult. The perpendiculars are the radii of the inscribed circle. (Page 40.)

22. Let A B D F and *a b d f* be two squares, one circum-



scribed about, and the other inscribed in a circle; let F D be 2, then the area of A B D F is 4, and of $a\ b\ d\ f$ 2 (6. and 7. IV. and I. 41.): thence it is required to find the areas of two other regular polygons of double the number of sides (eight), the one circumscribed and the other inscribed, viz., $h\ i\ k\ w\ v\ z\ y\ x$, and G f E d C b Q a.

First, it is evident that $o\ g\ f$ and o G F are similar, and are also like portions of the squares $a\ b\ d\ f\ a$, and A B D F A; it is also evident that the triangle o G f, and the quadrilateral o G $h\ f$, are like portions of the inscribed and circumscribed polygons, of which G f and $h\ i$ are sides; and as $g\ f$ is parallel to G F, $o\ g$: o G :: $o\ f$: o F; but $o\ g$: o G :: the triangle $o\ g\ f$: o G f (1. VI.); for the same reason $o\ f$: o F :: the triangle o G F : the triangle o G F; therefore, by equality of ratios, the triangle $o\ g\ f$: o G f :: o G f : o G F; therefore the polygon G f E d C b Q a is a mean proportional between $a\ b\ d\ f\ a$ and A B D F A; but $a\ b\ d\ f$ is 2, and A B D F is 4; therefore, G f E d C b Q a is equal $\sqrt{2 \times 4} = 2{\cdot}8284271$.

Again, because $o\ h$ bisects the angle G o F, G o : o F, or $g\ o$: $o\ f$ :: G h : h F (3. VI.) but $g\ o$: $o\ f$ :: $g\ o$: o G :: the triangle $o\ g\ f$: o G f; and $o\ g$: $o\ f$ (= o G) :: o G : o F :: G h : h F :: the triangle o G h : the triangle $o\ h$ F; then from equality of ratios the triangle $o\ g\ f$: the triangle o G f :: o G h : the triangle $o\ h$ F. Hence the inscribed figure $a\ b\ d\ f$ is to its derivative inscribed figure G f E d C b Q a as the triangle o G h to the triangle $o\ h$ F; and as magnitudes are proportionals by conversion, and also by the similar increase of their homologous terms, it follows that $a\ b\ d\ f$ and G f E d C b Q a together are to twice $a\ b\ d\ f$ as the triangles o G h and $o\ h$ F together (= o G F) to twice the triangle o G h (= o G $h\ f$); that is, $a\ b\ d\ f$ together with G f E d C b Q a is to A B D F as o G F to o G $h\ f$; but o G F and o G $h\ f$ are like portions of A B D F and $x\ h\ i\ k\ w\ v\ z\ y$; therefore, $a\ b\ d\ f$ and G f E d C b Q a together are to A B D F as A B D F to $x\ h\ i\ k\ w\ v\ z\ y$. But $a\ b\ d\ f = 2$, and G f E d C b Q $a = 2{\cdot}8284271$; hence, $2 + 2{\cdot}8284271 : 4 :: 4$ (= A B D F) : $3{\cdot}3137085$ = the area $x\ h\ i\ k\ w\ v\ z\ y$.

From this it appears that the two inscribed polygons are

to twice the primitive inscribed polygon, as the area of the circumscribed polygon, to the area of the derivative circumscribed polygon having double the number of sides. Hence, to find the area of the inscribed polygon of 16 sides, which is a mean proportional between 2·8284271 and 3·3137085, we multiply them together, and extract the square root of the product, thus, $\sqrt{(2{\cdot}8284271 \times 3{\cdot}3137085)} = 3{\cdot}0614674$; and to find the area of the circumscribed polygon of 16 sides, we say, as $2{\cdot}8284271 + 3{\cdot}0614674 : 2 \times 2{\cdot}8284271$, or, $5{\cdot}8898945 : 5{\cdot}6568542 :: 3{\cdot}3137085 : 3{\cdot}1825979$.

Pursuing this mode of calculation, namely, extracting the square root, and finding a fourth proportional alternately, the following table may be formed, showing the numbers expressing the areas of the inscribed and circumscribed polygons.

Number of sides.	Areas of the inscribed Polygon.	Area of the circumscribing Polygon.
5	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

As the area of the circle is intermediate between the areas of the inscribed and circumscribed polygons, and as the areas of the last two in the table are the same in all the figures to

which they are carried out, it follows that the area of the circle must also be 3·1415926.

Now, if we conceive the circumscribed polygon to consist of an infinite number of sides, the sum of all is evidently equal to the circumference of the circle; but the area of such a polygon is found by multiplying half the sum of all the sides by the radius; therefore the area of a circle is found by multiplying half the circumference by the radius; but as the area to the radius 1, is 3·1415926, or 3·1416 nearly, it is evident that 3·1416 is half the circumference of a circle whose radius is 1, and the circumferences of circles being as their radii, 3·1416 is the circumference of a circle whose diameter is 1. Hence, if D denote the diameter of any circle, its circumference will be found by the analogy, 1 : 3·1416 :: D : the circumference; the converse of this is evident.

Smaller numbers expressing the approximate ratio of 1 to 3·1416 are 113 to 355 and 7 to 22. These numbers are found by continued fractions.

To find the area, having the diameter.—Since the areas of circles are to each other as the square of their diameters, it will be, as $2^2 : D^2 :: 3·1416$: the area of the circle whose diameter is D, this $= \frac{D^2}{2} \times 3·1416 = D^2 \times ·78539815$, or $D^2 \times ·7854$.

The diameter and circumference of a circle are incommensurable; and though Geometry furnishes no method of finding a line equal to the circumference, yet the approximate numerical solution given above answers all the purposes to which it may be applied.

From what has been said, it is easy to conceive that the area of a circle is equal to the area af a right-angled triangle, whose altitude is equal to the radius of a circle, and base equal to its circumference. (Page 41.)

23. It has been shown that, when the radius is 1, the semi-circumference is 3·1416, which being divided by 180,

the degrees in a semi-circle gives ·01745329, the length of 1 degree, to the radius 1. (Page 42.)

24. Let the radius C D $= r$, and sine A P $=$ S; then the chord A D $= \surd\,(S^2 + [r - \surd\,(r^2 - S^2)]^2) = S + \frac{S^3}{8\,r^2} + \frac{7\,S^5}{128 r^4}$ $+$ &c.; and 8 A D $+ 8\,S + \frac{S^3}{r^2} + \frac{7\,S^5}{16 r^4} +$ &c.; therefore, 8 A D $-$ 2 A P $= 6\,S + \frac{S^3}{r^2} + \frac{7\,S^5}{16 r^4}$ &c.; and $\frac{1}{3}$(8 A D $-$ 2 A P) $= \frac{1}{3}$(8 A D $-$ A B) $= 2\,S + \frac{S}{3\,r^2} + \frac{7\,S^5}{48\,r^4}$, &c. But from Trigonometry, the length of the arc A D, whose sine is S, is known to be $S + \frac{S^3}{6\,r^2} + \frac{3\,S^5}{40\,r^4}$, &c.; and therefore, the arc A D B will be $2\,S + \frac{S^3}{3\,r^2} + \frac{6\,S^5}{40\,r^4}$, &c. Now when we compare this expression for the length of the arc with $\frac{1}{3}$ (8 A D $-$ A B) $= 2\,S + \frac{S^3}{3\,r^2} + \frac{7\,S^5}{48 r^4}$, &c., we find the difference to be only $\frac{S^5}{240\,r^4}$ which proves the truth of the rule. (Page 42.)

The reason of the rule may be otherwise shown thus:—Let arc A D B $= 2$ A, chord A D $= c$, A P $=$ S; then, by Trigonometry, A $= S + \frac{S^3}{2.3\ r^2}$, &c., and 2 A $= 2\,S + \frac{S^3}{3.r^2}$. Also $\frac{A}{2} = \frac{c}{2} + \frac{\left(\frac{c}{2}\right)^3}{2.3r^2}$, &c.; then A $= c + \frac{c^3}{8.3r^2}$; and 8 A $= 8c + \frac{c^3}{3r^2}$, &c.; hence 8 A $-$ 2 A $=$ 6 A $= 8\,c + \frac{c^3}{3r^2}$, &c. $-$ 2 S

$-\frac{S^3}{3r^2}$, &c., but $\frac{c^3}{3\,r^2}$ being nearly equal to $\frac{S^3}{3\,r^2}$; therefore, $6\,A = 8\,c - 2\,S$, and $2\,A\,(= A\,D\,B = \frac{8\,c - 2\,S}{3}$, which is the rule.

The following formula taken from Dr. Hutton is more accurate :—

Let d = the diameter, and v the versed sine, then $(5\,d\,\sqrt{\frac{5\,v}{5\,d-3\,v}} + 4\,\sqrt{d\,v}) \times \frac{2}{9}$ = the length of the arc.

Demonstration. From Trigonometry we have $A = S + \frac{S^3}{2.3.r^2} + \frac{3\,S^5}{2.4.5.r^4} + \frac{3.5.S^7}{2.4.6.7.r}$, &c., that is, $A = S \times (1 \times \frac{S^2}{2.3.r^2} + \frac{3\,S^4}{2.4.5.r^4} + \frac{3.5.S^6}{2.4.6.7.r^6}$, &c.)

The length of the curve A D B being required, call its half D B A, the versed sine D P, v, sine B P, S, and radius r; and because $K\,P \times P\,D = B\,P^2$; that is, $(2\,r - v) \times v = S^2$, and then $\sqrt{(2\,r\,v - v^2)} = T$; substituting this value of S in the above equation, we get $A = \sqrt{(2\,r\,v - v^2)} \times \left(1 + \frac{2rv - v^2}{2.3.r^2} + \frac{3 \times (2\,r\,v - v^2)^2}{2.4.5.r^4} + \frac{3.5 \times (r\,v - v^2)^3}{2.4.6.7.r^6}, \&c.\right)$

By expanding and multiplying these factors, we get

$$A = \sqrt{2\,r\,v} \times (1 + \frac{v}{2.3.2r} + \frac{3\,v^2}{2.4.5.2^2 r^2} + \frac{3.5.\,v^3}{2.4.6.7.2^3\,r^3}, \&c.)$$

Put $2\,r = d$, and we get

$$A = \sqrt{d\,v} \times (1 + \frac{v}{2.3\,d} + \frac{3\,v^2}{2.4.5.\,d^2} + \frac{3.5\,v^3}{2.4.6.7\,d^3}, \&c.)$$

To find the value of this series, let it be assumed $= d\,\sqrt{\frac{v}{g\,d - h\,v}} + n\,\sqrt{d\,v}$, and we get

$$A = d\sqrt{\frac{v}{g\,d - h\,v}} + n\sqrt{d\,v} = \sqrt{\frac{d^2\,v}{g\,d - h\,v}} + n\sqrt{d\,v} = \sqrt{d\,v} \times \left(\sqrt{\frac{d}{g\,d - h\,v}} + n\right) = \sqrt{d\,v} \times \left(\sqrt{\left(\frac{1}{g - \frac{h\,v}{d}}\right)} + n\right) = \sqrt{d\,v} \times \frac{1}{\left(\sqrt{\left(g - \frac{h\,v}{d}\right)} + n\right)}$$

$$= \sqrt{d\,v} \times \frac{1}{\left(\left(g - \frac{h\,v}{d}\right)^{\frac{1}{2}} + n\right)} \quad \sqrt{d\,v} \times \left(\left[g - \frac{h\,v}{d}\right]^{-\frac{1}{2}} + n\right).$$

This being expanded, we get

$$\sqrt{d\,v} \times \left(\frac{1}{g^{\frac{1}{2}}} + \frac{h\,v}{2\,d\,g^{\frac{3}{2}}} + \frac{3\,h^2\,v^2}{8\,h^2\,g^{\frac{5}{2}}}, \text{\&c.} + n\right)$$

$$= \sqrt{d\,v} \times \left(\frac{1}{g^{\frac{1}{2}}} + n + \frac{h\,v}{2\,d\,g^{\frac{3}{2}}} + \frac{3\,h^2\,v^2}{8\,d^2\,g^{\frac{5}{2}}}, \text{\&c.}\right)$$

By comparing with the equation,

$$A = \sqrt{d\,v} + \left(1 + \frac{v}{2.3\,d} + \frac{3\,v^2}{2.4.5\,d^2} + \frac{4.5\,v^3}{2.4.6.7\,d^3}, \text{\&c.}\right)$$

we get the co-efficients of the like powers of v equal to each other; that is, $\frac{1}{g^{\frac{1}{2}}} + n = 1$; then, $n = 1 - \frac{1}{g^{\frac{1}{2}}} = \frac{g^{\frac{1}{2}} - 1}{g^{\frac{1}{2}}}$ and $\frac{1}{6} = \frac{h}{2\,g^{\frac{3}{2}}}$, and $\frac{1}{3} = \frac{h}{g^{\frac{3}{2}}}$; hence, $\frac{g^{\frac{3}{2}}}{3} = h$, and $\frac{g^3}{9} =$

h^2. Again, we have $\frac{3}{40} = \frac{3\,h^2}{8\,g^{\frac{5}{2}}}$; hence, $\frac{1}{5} = \frac{h^2}{g^{\frac{5}{2}}}$; and $\frac{g^{\frac{5}{2}}}{5}$ $= h^2$; therefore, $\frac{g^{\frac{5}{2}}}{5} = \frac{g^3}{9}$; hence, $g^3 = \frac{9}{5} \times g^{\frac{5}{2}}$; that is, $g^{\frac{6}{2}} = \frac{9}{5} \times g^{\frac{5}{2}}$, and dividing by $g^{\frac{5}{2}}$, we get $g^{\frac{1}{2}} = \frac{9}{5}$, and $g = \frac{81}{25}$; but $h^2 = \frac{g^{\frac{5}{2}}}{5} = \frac{\left(\frac{81}{25}\right)^{\frac{5}{2}}}{5} = \left[\frac{81}{25}\right]^2 \times \frac{9}{25}$; hence, $h = \frac{81}{25} \times \frac{3}{5}$, and substituting those in the equation, $A = d \sqrt{\frac{v}{g\,d - h\,v}} + n \sqrt{d\,v}$, we get $A = d \sqrt{\frac{v}{\left(\frac{81}{25}\,d - \frac{81}{25} \times \frac{3}{5}\,v\right)}} + \frac{4}{9} \sqrt{d\,v}$; (for $n = \frac{g^{\frac{1}{2}} - 1}{g^{\frac{1}{2}}} = \frac{\sqrt{\left(\frac{81}{25}\right)} - 1}{\sqrt{\left(\frac{81}{25}\right)}} = \frac{4}{9}$) $= d \sqrt{\frac{v}{(d - \frac{3}{5}v) \times \frac{81}{25}}} + \frac{4}{9} \sqrt{(d\,v)}$ $= \frac{d}{\frac{9}{5}} \sqrt{\left(\frac{v}{d - \frac{3}{5}v}\right)} + \frac{4}{9} \sqrt{d\,v} = \frac{5\,d}{9} \sqrt{\frac{v}{d - \frac{3}{5}\,v}} + \frac{4}{9}$ $\sqrt{d\,v} = \left(5\,d \sqrt{\frac{5\,v}{5\,d - 3\,v}} + 4 \sqrt{d\,v}\right) \times \frac{1}{9}$; and $2\,A =$ $\left(5\,d \sqrt{\frac{5\,v}{5\,d - 3\,v}} + 4 \sqrt{d\,v}\right) \times \frac{2}{9}$, which is the rule.

25. When the circumference of a circle is 1, its diameter is, by Prob. XVI., found to be ·318309, and its area is

$\frac{\cdot 318309}{2} \times \frac{1}{2}$, (Demonstration 22.) $= \frac{\cdot 318309}{4} = \cdot 079577 =$ $\cdot 07958$ nearly; but circles are to each other as the squares of their circumferences; therefore, $1^2 : \text{cir.}^2 :: \cdot 07958 : \text{area} = \text{cir.}^2 \times \cdot 07958$.

Rules IV. and V. $7 : 22 ::$ diameter $1 :$ circumference $= \frac{22}{7}$, and the area is $\frac{22}{7 \times 2} \times \frac{1}{2} = \frac{11}{14}$; hence, $14 : 11 :: d^2 :$ area; $22 : 7 ::$ circumference $1 :$ diameter $= \frac{7}{22}$ and $\frac{7}{22 \times 2} \times \frac{1}{2} = \frac{7}{88}$ the area of a circle whose circumference is 1; therefore, $1^2 : \text{cir.}^2 :: \frac{7}{88} :$ area; that is, $88 : 7 : \text{cir.}^2 :$ area cir. being the circumference. (Page 43.)

Cor. Hence, if $d =$ diameter, $c =$ circumference, $a =$ area, $n = 3{\cdot}1416$. Then,

1st. $d = \frac{c}{n} = \frac{4\,a}{c} = \sqrt{\left(\frac{a}{n}\right)}$. For $n : 1 :: c : d = \frac{c}{n}$ and $\frac{d}{2} \times \frac{c}{2} = a$; then, $c\,d = 4\,a$, and $d = \frac{4\,a}{c}$; again, $1 : n :: d : n\,d = c$; then, $\frac{4\,a}{c} = \frac{4\,a}{n\,d} = d$; then, $d^2 = \frac{4\,a}{n}$, and $d = \sqrt{\left(\frac{4\,a}{n}\right)} = 2\sqrt{\left(\frac{a}{n}\right)}$.

2nd. $c = n\,d = \frac{4\,a}{d} = 2\sqrt{(n\,a)}$. For $c\,d = 4\,a \therefore c = \frac{4\,a}{d}$; again, $c = \frac{4\,a}{d} = \frac{4\,a}{\frac{c}{n}} = \frac{4\,a\,n}{c} \therefore c^2 = 4\,a\,n$, and $c = 2\sqrt{(a\,n)}$.

3rd. $a = \frac{n\ d^2}{4} = \frac{c^2}{4\ n} = \frac{d\ c}{4}$. For $c = n\ d \therefore \frac{d}{2} \times \frac{n\ d}{2}$ $= a = \frac{n\ d^2}{4}$, and $c^2 = 4\ a\ n \therefore a = \frac{c^2}{4\ n}$; likewise, $c\,d =$ $4\ a$; hence $a = \frac{c\,d}{4}$.

4th. $n = \frac{c}{d} = \frac{4\ a}{d^2} = \frac{c^2}{4\ d}$. For $c = n\ d \therefore n = \frac{c}{d}$, and $a = \frac{n\ d^2}{4}$; then, $4\ a = n\ d^2 \therefore n = \frac{4\ a}{d^2}$; again, $d^2 = \frac{4\ a}{n}$ $\therefore n\ d^2 = 4\ a$, and $n = \frac{4\ a}{d^2}$.

26. When the diameter is 1, the area is ·7854 (Demonstration 22); then the side of a square equal in area to ·7854 is $\sqrt{}$ (·7854) $=$ ·8862269, the multiplier. (Page 44.)

27. When the circumference is 1, the area of the circle is ·079577 (Demonstration 25); then, the side of the square equal in area to ·079577 is $\sqrt{}$ (·079577 $=$ ·2820948, the multiplier. (Page 45.)

28. When the diameter is 1, ($=$ B D), the area of the circumscribed square is 1, and therefore the area of the inscribed square is $\frac{1}{2}$ ($=$ ·5) (Demonstration 22), and the side itself is $\sqrt{}$ ·5 $=$ ·7071068. (Page 45.)

29. Let the area of the circle A B G D be 1, then, from the first equation in the Cor. to Demonstration 25 $d = 2 \sqrt{\left(\frac{1}{3{\cdot}1416}\right)}$, and $\frac{d}{2} = \sqrt{\left(\frac{1}{3{\cdot}1416}\right)}$; hence, A o $= \sqrt{\left(\frac{1}{3{\cdot}1416}\right)}$, and A $o^2 = \frac{1}{3{\cdot}1416}$; but A $o^2 + o\,\mathrm{D}^2 =$

$2\ \mathrm{A}\ o^2 = (47.\ \mathrm{I.}) = \mathrm{A\ D}^2 = \dfrac{2}{3{\cdot}1416} = {\cdot}6366197.$ Now, similar figures are as the squares of their like sides, $\therefore$ 1 : given area :: ·6366197 ($= \mathrm{A\ D}^2$) to the square of the side of the inscribed square = given area $\times$ ·6366197 $\therefore$ the side itself is $\sqrt{}$ (given area $\times$ ·6366197), which is the rule. (Page 45.)

30. When the side of the square is 1, the radius of the circumscribed circle is ·7071068, and the diameter is ·7071068 $\times$ 2 = 1·4142136 —See Table I. (Page 46.)

31. When the side of the square is 1, the diameter of the circumscribed circle is 1·4142136; and therefore, its circumference 1·4142136 $\times$ 3·1416 = 4·4428934. Hence the reason of the rule. (Page 46.)

32. From the first equation, in the Cor. to Demonstration 25, we have $d = \sqrt{\dfrac{4\,a}{n}}$; but when the side of a square is 1, its area is 1 $\therefore d = \sqrt{\left(\dfrac{4}{n}\right)} = \sqrt{\left(\dfrac{4}{3{\cdot}1416}\right)} = 1{\cdot}1283791.$ Hence the reason of the rule. (Page 46.)

33. By the last Demonstration, the diameter of a circle equal in area to a square, whose side is 1, is 1·1283791, and its circumference, therefore, will be 1·1283791 $\times$ 3·1416 = 3·5449076. (Page 47.)

34. The reason is evident from Demonstration 22. (Page 47.)

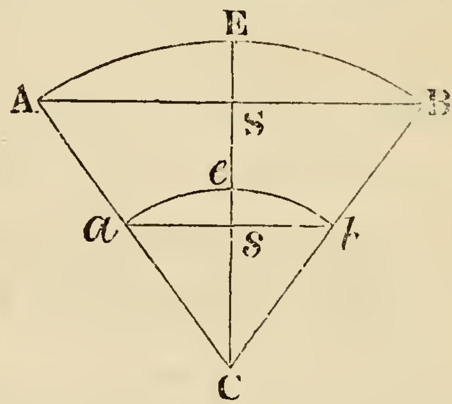
35. The reason of the second rule is evident from the 33. VI., which proves that arcs of the same circle are as the sectors; therefore the whole circumference (360) is to the given arc, as the area of the whole circle is to the area of the sector. (Page 47.)

36. This rule, which is the best for practice, was originally demonstrated by Mr. Peter Nicholson. He, however, took the fundamental part of his demonstration from Dr. Hutton, which the Doctor discovered by Fluxions. Let v = versed sine C D of the arc A C B and d the diameter of the circle; then, the length of half the arc is $\sqrt{(d\, v)} \times \left(1 + \frac{v}{6\, d} + \frac{3\, v^2}{40\, d^2}, \text{\&c.}\right)$ (see page 69, Appendix;) this multiplied by half the diameter, will give the area of the sector; that is, the area of the sector is

$\frac{d}{2} \sqrt{(d\, v)} \times \left(1 + \frac{v}{6\, d} + \frac{3\, v^2}{40 d^2}, \text{\&c.}\right)$ Now, it is easy to conceive that $\pm \frac{1}{2}\, d \mp v$ = the altitude of the triangle whose base A B is $2 \sqrt{(d . v - v^2)} = 2 \sqrt{(d\, v)} \times \left(1 - \frac{v}{2\, d} - \frac{v^2}{8\, d^2}, \text{\&c.}\right)$ Hence, the area of the triangle is $(\pm \frac{1}{2}\, d \mp v) \times \sqrt{(d\, v)} \times \left(1 - \frac{v}{2\, d} - \frac{v^2}{8\, d^2}, \text{\&c.}\right)$ which being added to, or subtracted from the area of the sector, gives $2\, v \sqrt{(d\, v)} \times \left(\frac{2}{3} - \frac{v}{5\, d} - \frac{v^2}{28\, d^2}, \text{\&c.}\right)$ But D F $\times$ D C = A D² (35. III.); that is, $(d - v) \times v = \frac{c^2}{4}$, ($c$ being put for A B); therefore, $d = \frac{c^2}{4\, v} + v$, and $2\, v \sqrt{(d\, v)} = v \sqrt{(c^2 + 4\, v^2)} = v\, c + \frac{2\, v^3}{c} - \frac{2\, v^5}{c^3} + \frac{4\, v^7}{c^5}$, &c. by extracting the square root of $c^2 + 4\, v^2$, and multiplying by v); therefore, $2\, v \sqrt{(d\, v)} \times \left(\frac{2}{3} - \frac{v}{5\, d} - \frac{v^2}{28\, d^2}\right) = \left(v\, c + \frac{2\, v^3}{c}\right) \times \left(\frac{2}{3} - \frac{v}{5\, d}\right) = \frac{2}{3}\, v\, c + \frac{4\, v^3}{3}\, c -$

$\frac{v\,c^2}{5\,d}$, &c. $= \frac{2}{3}\,v\,c + \frac{4\,v^3}{3\,c} - \frac{4\,c\,v^3}{5\,c^2 + 20\,v^2} = \frac{2}{3}\,v\,c + \frac{8\,c^2\,v^3 + 80\,v^5}{15\,c^3 + 60\,c\,v^2} = \frac{2}{3}\,v\,c + \frac{8\,v^3}{16\,c}$, or $\frac{2}{3}\,v\,c + \frac{v^3}{2\,c}$ nearly.— (Page 50.)

37. To prove the truth of this rule, it will be necessary to show that segments, whose versed sines are as the diameters, will be to each other as the squares of the diameters. Let A E B A and *a e b a* be two similar segments, cut from the similar sectors, A E B C A and *a e b c a* by the chords A B and *a b*. Draw C E, bisecting both the arcs.



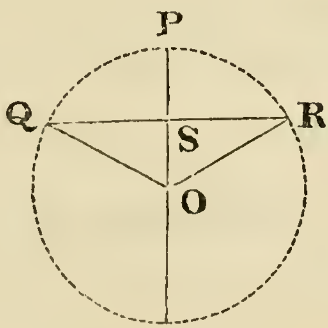
By similar triangles, C A : C *a* :: C S : C *s*; that is, C A : C *a* :: C A — E S : C *a* — *e s* ∴ C A : C *a* :: E S : *e s*. Hence, the versed sines of similar segments are as the radii of the circles, or as the diameters; but similar sectors are as the squares of the diameters, and similar triangles as the squares of their like sides; C A^2 : C a^2 :: sector A E B C A : sector *a e b* C *a* :: triangle C A B : triangle C *a b* :: seg. A E B A (= sector A E B C A — triangle A B C) : seg. *a e b a* (= sector *a e b* C *a* — triangle

$a\ b$ C); that is, the segments are to each other as the squares of the diameters.

Now the diameter in the tables is 1, then, by putting $d =$ any diameter, and $v =$ versed sine, we shall have $d : v :: 1 : v \div d =$ versed sine of a similar segment in the table, whose area we shall call a. Then, from what has been proved, $1^2 : d^2 :: a : a\ d^2 =$ area of the segment, whose height or versed sine is v, and diameter d. (Page 51.)

Note.—If to the square of half the chord of the segment there be added the square of the versed sine, the square root of the sum will give the chord of half the arc of the segment. To $\frac{4}{3}$ of the chord of half the arc of the segment add the chord of the segment, the sum multiplied by $\frac{2}{5}$ of the versed sine will give the area. The truth of this rule may be shown thus:—

As in Dem. 36. we have $A = \surd\ (d\ v) \times \left(1 + \frac{v}{2.3\ d}\right.$



$\left.+ \frac{3\ v^2}{2.4.5\ d^2}, \&c.\right)$; and therefore, the area of the sector Q P R O is $\frac{d}{2} \times \left(\surd\ (d\ v) \times (1 + \frac{v}{2.3\ d} + \frac{3\ v^2}{2.4.5\ d^2}\right)$ but Q S $\times$ S O $=$ area of the triangle Q O R; that is, $\surd\ ((d - v) \times v) \times \left(\frac{d}{2} - v\right) = \surd\ (d\ v - v^2) \times \left(\frac{d}{2} - v\right)$

$=$ area of the triangle Q O R; and this expanded, and taken from the area of the sector, as found before, we get $2v \times \left(\surd (d\,v) \times \left(\frac{2}{3} - \frac{v}{5\,d} - \frac{v^2}{28\,d^2}, \text{\&c.}\right)\right)$ for the area of the segment in terms of v and d, which assume equal to $2\,v\,(m \surd (d\,v - v^2) + n \surd (d\,v))$ (in order to find a finite value for the segment) $= 2\,v\,(m \surd \left(d\,v \times (1 - \frac{v}{d})\right) + n \surd (d\,v)) = 2\,v\,(m \surd (d\,v) \times \left(1 - \frac{v}{d}\right)^{\frac{1}{2}} + n \surd (d\,v)) = 2\,v\,(\surd (d\,v) \times (m \times \left(1 - \frac{v}{d}\right)^{\frac{1}{2}} + n) = 2\,v \surd (d\,v) \times [m \times \left(1 - \frac{v}{2\,d} - \frac{v^2}{8\,d^2}\right) + n] = 2\,v \times \surd (d\,v) \times \left(m - \frac{m\,v}{2\,d} - \frac{m\,v^2}{8\,d^2}, \text{\&c.} + n\right) = 2\,v \times \surd (d\,v) \times \left(m + n - \frac{m\,v}{2\,d} = \frac{m\,v^2}{8\,d^2}, \text{\&c.}\right)$ Now, by comparing these two expressions for the area of the segment, we get $m + n = \frac{2}{3}$ $-\frac{m}{2} = -\frac{1}{5}$; therefore, $m = \frac{2}{5}$; then, $\frac{2}{5} + n = \frac{2}{3}$, and $n = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$. By substituting these in the equation expressing the segment, we get $2\,v \times \frac{2}{5} \surd (d\,v) - v^2) + \frac{4}{15} \surd (d\,v) = \frac{2}{5}\,v \times (2 \surd (d\,v - v^2) + \frac{4}{3} \surd (d\,v)$; but $\surd (d\,v)$ is the chord of half the arc, and $2 \surd (d\,v - v^2)$ is the chord of the segment, and $\frac{2}{5}\,v$ is $\frac{2}{5}$ of the versed sine; therefore, $\frac{2}{5}\,v \times \left(2 \surd (d\,v - v^2) + \frac{4}{3} \surd (d\,v)\right)$ expresses the rule.

38. The triangles A x F and D x B being similar; D x : D B :: A x : A F, which is double G y; then, G y = D B $\times$ A $x \div 2$ D x.

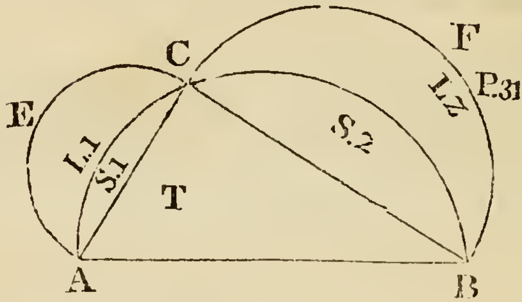
When the diameter of the circle is given, or the chord A C, and height $y\,z$, the operation is very simple, in which case D x needs not be given. (Page 52.)

39. Let D be the diameter of the larger circle, and d the diameter of the smaller; then $D^2 \times \cdot 78554$ = area of the larger circle, and $d^2 \times \cdot 7854$ = area of the smaller circle $\therefore D^2 \times \cdot 7854 - d^2 \times \cdot 7854 = (D^2 - d^2 \times \cdot 7854 = (D + d) \times (D - d \times \cdot 7854$ = the area of the ring. And this expression corresponds with the rule. (Page 53.)

40. Let A B = A, and $a\,b = a$; let, also, A C, the radius, $= r$. Then, as similar arcs are to each other as their radii, A B : $a\,b$ :: C A : C a; that is, $A : a :: r : \frac{a\,r}{A} = a\,C$; but $A\,C - a\,C = a\,A = r - \frac{a\,r}{A} = \frac{A\,r - a\,r}{A} = \frac{(A - a)}{A} \times r$. But the area of the sector A C B $= \frac{A\,r}{2}$ and the area of the sector a C b $= \frac{C\,a \times a\,b}{2} = \frac{a\,r}{A} \times \frac{a}{2} = \frac{a^2\,r}{2\,A}$ $\therefore$ the area of the segment A B $b\,a$ $= \frac{A\,r}{2} - \frac{a^2\,r}{2\,A} = \frac{A^2\,r - a^2\,r}{2\,A} = \frac{(A^2 - a^2)}{2\,A} \times r = \frac{(A + a)}{2} \times \frac{(A - a)}{A} \times r = \frac{A + a}{2} \times A\,a$, by substituting A a for its equal $\frac{A - a}{A} \times r$, which is the rule. (Page 53.)

Remark on Prob. XXXII.

If A B C be a right-angled triangle, on the three sides of which, if three semi-circles be described; then, the triangle



T (A B C) will be equal to the sum of the two lunes, L 1, L 2. Because the sum of the semi-circles described on the sides containing the right angle is equal to the semi-circle described on the hipothenuse, (2. XII., 24. V., and 47. I.,) and taking away the segments, S 1, S 2, which are common to the equal quantities, the remainders will be equal, viz., the sum of the lunes, L 1, L 2, will remain equal to T. (Page 54.)

41. This rule is evident from Proposition XIII. Cor. 2. Ellipsis.* (Page 61.)

42. The first rule is self-evident; for the space E G F H being deducted from A C B D; the remainder will be the space between both circumferences.

By putting T and C for the transverse and conjugate diameters of the larger ellipse, its area is T $\times$ C $\times$ ·7854, and the area of the smaller ellipse is $t \times c \times$ ·7854; then, their difference is T $\times$ C $\times$ ·7854 — $t \times c \times$ ·7854 = (T $\times$ C — $t \times c$) $\times$ ·7854, which is the second rule. (Page 62.)

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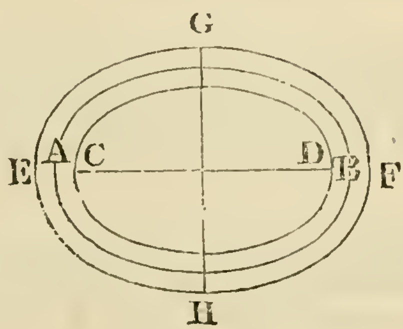
* This and similar references are to the Appendix, Properties of the Conic Sections.

43. By Corollaries to Proposition XIII. Ellipsis, an ellipsis is to the rectangle of its two axis, as any circle is to the square of its diameter. And also any segment of an ellipse is to a like segment of a circle, as the rectangle contained by the two axes of the ellipse is to the square of the diameter of the circle. But by Prob. XXVIII. Rule 3, the segment of the circle is found by multiplying by the square of the diameter, the area segment, as found in the Table of circular segments, corresponding to the height, divided by the diameter: therefore, the segment of the ellipse is equal to the product of both its axes, multiplied by the area segment corresponding to the height of the segment divided by the diameter, of which the given height is a part. (Page 63.)

44. It has been shown that the geometrical mean between the two axes is equal to the diameter of a circle equal in area to the ellipse; but the arithmetical mean exceeds this geometrical mean, while the circumference of the ellipse exceeds that of the circle equal in area to it; therefore, $\frac{(t + c)}{2} \times p = (t + c) \times \frac{1}{2} p$ will give the circumference nearly.



$\frac{(t \times c)}{2} \times p$ is greater than the circumference of a circle; but $\sqrt{(t \times c)} \times p$ is less than the circumference of the ellipse equal in area to the circle; therefore, $(t + c) \times \frac{1}{2} p$ is the circumference of the ellipse nearly.

Otherwise :—

Let A B be the curve whose length we require; and let two other curves, E G F and C D be described, at equal, but very small distances from it, the one without it, and the other within it; these two curves do not differ much from ellipses, and the difference of their areas, as found by Prob. II. Sec. III., will give the area of the ring or space between them nearly; but this area is equal to the curve A B multiplied by E C; the distance between the two curves E F and C D; therefore, the area of the ring divided by E C will give the length of the curve A B nearly enough for practical purposes.

Therefore, putting $t =$ the transverse axis of the ellipse, whose length is required,

$c =$ its conjugate
$p = 3{\cdot}1416$
$d = \frac{1}{2}$ E C $=$ A C, or A E.

Then, $(t + 2\,d) \times (c + 2\,d) \times \frac{1}{4}\,p\ (= {\cdot}7854) =$ the area of the curve E F.

And $(t - 2\,d) \times (c - 2\,d) \times \frac{1}{4}\,p =$ the area of the curve C D.

The difference $(t + c) \times p\,d =$ the area of the ring.

Therefore, $(t + c) \times p\,d \div 2\,d =$ the length of the curve A B; that is, $(t + c) \times \frac{1}{2}\,p =$ the length of the curve. (Page 64.)

45. It is manifest that the circular arc $x\ y$ is an arithmetical mean between the circular arcs C V and G B $\therefore$ $x\,y$ is nearly equal to the elliptical arc C B. Hence, the rule is evident.

This rule is the easiest for practice. Other rules might be given, which would find more accurate results; but being both tedious and difficult, and the investigation necessarily involving the fluxonial or differential calculus, they are omitted. (Page 64.)

46. The equation of the curve is $d^2 : c^2 :: x\,(d - x) : y^2$; putting d for the transverse diameter, c for its conjugate, and x for an abscissa to the ordinate y; then, $d : c :: \sqrt{(x\,(d - x))} : y$, which is the rule. (Page 66.)

47. From the property of the curve, we have $d^2 : c^2 :: x\,(d - x) : y^2$; then, $c^2\,d\,x - c^2\,x^2 = d^2\,y^2$, and dividing c^2, we get $d\,x - x^2 = \frac{d^2\,y^2}{c^2}$, and $x^2 - d\,x = -\frac{d^2\,y^2}{c^2}$ and by completing the square, we get $x^2 - d\,x + \frac{d^2}{4} = \frac{d^2}{4} - \frac{d^2\,y^2}{c^2} = d^2\left(\frac{1}{4} - \frac{y^2}{c^2} = d^2\left(\frac{1}{4}\,\frac{c^2 - y^2}{c^2}\right) = \frac{d^2}{c^2}\,(\frac{1}{4}\,c^2 - y^2)\right.$; then, $x - \frac{d}{2} = \pm\sqrt{\left(\frac{d^2}{c^2}\,(\frac{1}{4}\,c^2 - y^2)\right)} = \pm\frac{d}{c}\sqrt{(\frac{1}{4}\,c^2 - y^2)}$, and $x = \frac{d}{2} \pm \frac{d}{c}\sqrt{(\frac{1}{4}\,c^2 - y^2)}$, which corresponds with the rule. (Page 66.)

48. The same notation being retained, we have, from the property of the curve, $d^2 : c^2 :: (x\,d - x^2) : y^2$; and therefore, $y^2\,d^2 = c^2\,x\,d - c^2\,x^2$; then, by transposition, $y^2\,d^2 - c^2\,x\,d = -c^2\,x^2$, and by dividing both sides of the equation by y^2, we get $d^2 - \frac{c^2\,x}{y^2}\,d = -\frac{c^2\,x^2}{y^2}$, and by completing the square, we have $d^2 - \frac{c^2\,x}{y^2} + \frac{c^4\,x^2}{4\,y^4} = \frac{c^4\,x^2}{4\,y^4} - \frac{c^2\,x^2}{y^2}$ then, by extracting the square root of both these equal quantities, we get $d - \frac{c^2\,x}{2\,y^2} = \sqrt{\left(\frac{c^4\,x^2}{4\,y^4} - \frac{c^2\,x^2}{y^2}\right.}$; hence,

$$d = \frac{c^2 x}{2y^2} \pm \sqrt{\left(\frac{c^4 x^2}{4y^4} - \frac{c^2 x^2}{y^2}\right)} = \frac{cx}{y^2} \times [\tfrac{1}{2} c \pm \sqrt{(\tfrac{1}{4} c^2}$$

$- y^2)]$, which is the rule. For $\sqrt{(\frac{1}{4} c^2 - y^2)}$ is the square root of the difference of the squares of the semi-conjugate and ordinate, to which $\frac{1}{2} c$ is added for one factor; then, from the nature of proportion, $y^2 : c x :: \frac{1}{2} c \pm \sqrt{(\frac{1}{4} c^2 -}$

$y^2) : d = \dfrac{c x}{y^2} \times [\frac{1}{2} c \pm \sqrt{(\frac{1}{4} c^2 - y^2)}]$. (Page 66.)

49. From the equation of the curve, we have $d^2 : c^2 :: x \times (d - x) : y^2$; the roots of these are proportionals, viz. $\sqrt{[x \times (d - x)]} : y :: d : c$, which is the rule. (Page 67.)

50. This is proved in Proposition VII., Arith. of Infin., where it is shown that the area of a parabola is equal to $\frac{2}{3}$ of its circumscribed parallelogram. But the base, multiplied by the height, is the area of the circumscribed parallelogram; then, $\frac{2}{3}$ of this area is the area of the parabola. (Page 68.)

51. By Prop. III. Cor. 1. Parabola, $D\ O : x\ O :: A\ D^2 : S\ x^2$; and putting $A\ B = D$, $S\ T = d$, $D\ x = a$; $D\ O : x\ O \therefore \frac{1}{4} D^2 : \frac{1}{4} d^2 :: D^2 : d^2$; and by division, (17. V.) $D\ O - x\ O : x\ O :: D^2 - d^2 : d^2$; that is, $D\ x : x\ O :: D^2 - d^2 : d^2$; but $D\ x = a \therefore a : x\ O :: D^2 - d^2 :$

d^2. Hence, $x\ O = \dfrac{a\,d^2}{D^2 - d^2}$, and $D\ O = a + \dfrac{a\,d^2}{D^2 - d^2}$

But $D \times \left(a + \dfrac{a\,d^2}{D^2 - d^2}\right) \times \frac{2}{3}$ is equal the area of the

whole parabola, and $d \times \dfrac{a\,d^2}{D^2 - d^2} \times \frac{2}{3}$ is equal the area of

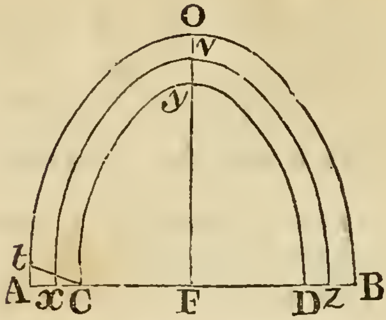
the segment S O T; their difference is the area of the zone A S T B, viz. :—

$(D \times \left(a + \frac{a\, d^2}{D^2 - d^2}\right) \times \frac{2}{3}) - \left(d \times \frac{a\, d^2}{D^2 - d^2}\right) \times \frac{2}{3} =$ $\frac{D^3 - d^3}{D^2 - d^2} \times \frac{2}{3}\, a = \frac{D^2 + D\, d + d^2}{D + d} \times \frac{2}{3}\, a$ is the area of the zone A S T B, which is the rule. (Page 69.)

52. Let $v =$ any curve beginning at the vertex O, $y =$ the ordinate to the axis at the extremity of the curve, $x =$ its abscissa, $a = \frac{1}{2}$ the parameter of the axis.

The equation of the curve, as shown in Prop. III. Cor. 2. Parabola, is $2\, a\, x = y^2$; the fluxion of these quantities will be equal; hence $2\, a\, \dot{x} = 2\, y\, \dot{y}$, and dividing both by $2\, a$, $\dot{x} =$ $\frac{y\, \dot{y}}{a}$; hence, $\dot{x}^2 = \frac{y^2 \dot{y}^2}{a^2}$, but $\dot{v} = \sqrt{\left(\dot{y}^2 + \dot{x}^2\right)}$; $\therefore\ \dot{v} =$ $\sqrt{\left(\dot{y}^2 + \frac{y^2 \dot{y}^2}{a}\right)} = \dot{y}\, \frac{\sqrt{(a^2 + y^2)}}{a}$. The fluents of these will be $v = y\, \frac{\sqrt{(a^2 + y^2)}}{2\, a} + \frac{1}{2}\, a \times$ hyp. log. of $y +$ $\left.\frac{\sqrt{(a^2 + y^2)}}{a}\right) = \frac{1}{2}\, a\, q \sqrt{(1 + q^2)} + \frac{1}{2}\, a \times$ hyp. log. of $[q + \sqrt{(1 + q^2)}]$, putting $q = \frac{y}{a}$; double of this quantity gives the value of the double curve, viz. $2\, v = c = a\, q \sqrt{(1 + q^2)} + a \times$ hyp. log. of $[q + \sqrt{(1 + q^2)}] = a \times (q\, S +$ hyp. log. of $q + S)$, which is the rule.

The length of the curve $x\, v\, y$ may be found pretty accurately thus:—Conceive two curves, A O B and C y D to be drawn equi-distant from the curve, whose length is required at a very small distance, say, the one hundred millionth part of a unit from it; then it is obvious that these two curves do not differ much from parabolas. Find the area of the parabola C y D to the ordinate C F, and height F y, which call S; find also the area of the parabola A O B to the ordinate A F, and height F O, which call G: then, G — S



will give the area of the ring A C y D B O A, which area being divided by the perpendicular distance O y, or the breadth of the ring, will give the length of a curve equal to a mean between the curve A O B and C y D; that is, of $x\,v\,z$; hence, $x\,v\,z = \frac{G - S}{y\,O}$. In this figure, O y = t C $= \frac{2}{1000000000}$ the mean breadth of the ring, from which A F and C F, may be determined, and hence G and S (Page 70.)

The fluxion of the curve in the last rule is $\dot{v} = \dot{y} \sqrt{\left(1 + \frac{y^2}{a^2}\right)}$ for $\dot{v} = \sqrt{\left(\dot{y}^2 + \frac{y^2 \dot{y}^2}{a^2}\right)} = \dot{y} \sqrt{\left(1 + \frac{y^2}{a^2}\right)}$ = by extracting the square root $\dot{y} \times (1 + \frac{y^2}{2\,a^2} - \frac{y^4}{2.4\,a^4} + \frac{3\,y^6}{2.4.6\,a^6}$, &c.); then, by finding the fluents, and putting $q = \frac{y}{a}$, we get

$v = y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3\,q^6}{2.4.6.7}$, &c.) = the length of one side of the curve, the double of which gives the entire length on both sides, viz. C $= 2\,v = 2\,y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3\,q^6}{2.4.6.7}$, &c.) (Page 70.)*

* By mistake no reference is made to this Demonstration in the Mensuration.

53. From what has been said in the last rule $v = y \times (1 + \frac{1}{2.3} q - \frac{1}{2.4.5} q^4 + \frac{3}{2.4.6.7} q^6$, &c.) and as $\sqrt{}(1 + \frac{1}{3} q^2) = 1 + \frac{1}{2.3} q^2 - \frac{1}{2.4.9} q^2 + \frac{3}{2.4.6.27} q^6$, &c.; hence $\frac{v}{y} - \sqrt{}(1 + \frac{1}{3} q^2) = - \frac{1}{2.5.9} q^4 + \frac{5}{2.7.27} q^6$, &c. which may be rejected, if we suppose q not greater than 1, $\therefore$ $v = y \sqrt{}(1 + \frac{1}{3} q^2) = \sqrt{}(y^2 + \frac{4}{3} x^2)$ nearly; $\therefore$ $C = 2v = 2\sqrt{}(y^2 + \frac{4}{3} x^2)$ nearly. (Page 71.)

54. This is evident from Proposition III. Cor. 1. Parabola. (Page 71.)

55. From the property of the parabola, D O : x O :: A D^2 : S x^2; then, by division, D O — x O (= D x) : x O :: A D^2 — S x^2 : S x^2; that is, A D^2 — S x^2 : S x^2 :: D x : x O. Also, A D^2 — S x^2 $\therefore$ A D^2 :: D x : D O. (Page 72.)

56. It is proved in Proposition V. Cor. 1. Hyperbola, that the square of half the transverse is to the square of half the conjugate, as the rectangle of the abscissa by the sum of the abscissa and transverse (which is the abscissa relatively to the opposite curve,) to the square of the ordinate therefore, $t^2 : c^2 :: x \times (t + x) : y^2$, and $t : c :: \sqrt{}(x \times (t + x)) : y$, which is the rule; t, c, x, y, being the transverse, conjugate, abscissa, and ordinate respectively— (Page 73.)

57. Retaining the same letters as in the last Demonstra-

tion, $\left(\frac{c}{2}\right)^2 : \left(\frac{t}{2}\right)^2 :: y^2 + \left(\frac{c}{2}\right)^2 : \left(\frac{t}{2} + x\right)$ (Prop. V. Cor. 4. Hyperbola); then, $c : t :: \surd\,(y^2 + \frac{c^2}{4}) : \frac{t}{2} + x =$ half the abscissas; then $\frac{t}{2} + x + \frac{t}{2} = t + x =$ greater abscissa, and $\frac{t}{2} + x - \frac{t}{2} = x \therefore \dfrac{t \surd \left(\frac{c^2}{4} + y^2\right)}{c} \pm \frac{t}{2} =$ the greater or less abscissa, which is the rule. (Page 74.)

58. From Proposition V. Cor. 1. Hyperbola, we have $\left(\frac{t}{2}\right)^2 : \left(\frac{c}{2}\right)^2 :: x \times (t + x) : y^2$; hence, $t^2 : c^2 :: x\,t + x^2 : y^2$; then, $y^2\,t^2 = c^2\,x\,t + c^2\,x^2$; and $y^2\,t^2 - c^2 x\,t = c^2\,x^2$; divide by y^2, and we get $t^2 - \frac{c^2 x}{y^2} \times t = \frac{c^2 x^2}{y^2}$ complete the square, and then, $t^2 - \frac{c^2 x}{y^2} \times t + \frac{c^4 x^2}{4\,y^4} = \frac{c^2 x^2}{y^2} + \frac{c^4 x^2}{4\,y^4} = \frac{4\,c^2\,x^2\,y^2 + c^4\,x^2}{4\,y^4}$, and $t - \frac{c^2 x}{2y^2} = \surd \left(\frac{4\,c^2\,x^2\,y^2 + c^4\,x^2}{4\,y^4}\right) = \surd \left[\frac{(4\,y^2 + c^2) \times c^2\,x^2}{4\,y^4}\right] = \frac{c\,x}{2y^2} \times \surd\,(4\,y^2 + c^2) \therefore t = \frac{c^2\,x}{2\,y^2} + \frac{c\,x}{2\,y^2} \surd\,(4\,y^2 + c^2) = \frac{c\,x}{2\,y^2} \times \surd\,(4\,y^2 + c^2) + c = \dfrac{c\,x \times \surd\,(y^2 + \frac{1}{4}\,c^2) + \frac{c}{2}}{y^2}$ which is the rule. (Page 75.)

59. By the last, we have $t^2 : c^2 :: (t + x) \times x : y^2$; then, the roots of these are proportionals; viz. $t : c :: \surd [(t + x) \times x] : y$; and then, $\surd [(t + x) \times x] : y :: t : c = \frac{t\,y}{\surd [(t + x) \times x]}$, which is the rule. (Page 75.)

60. Let X and x be the abscissas, and Y and y the corresponding ordinates; then, by Proposition V. Cor. 3. Hyperbola, we have $t\,x + x^2 : t\,X + X^2 :: y^2 : Y^2$; hence, $t\,X\,y^2 + X^2\,y^2 = t\,x\,Y^2 + x^2\,Y^2$; then, $t\,X\,y^2 - t\,x\,Y^2 = x^2\,Y^2 - X^2\,y^2$, and

$t = \frac{x^2\,Y^2 - X^2\,y^2}{X\,y^2 - x\,Y^2}$, or $t = \frac{X^2\,y^2 - x^2\,Y^2}{x\,Y^2 - X\,y^2}$, which is the rule. (Page 76.)

61. Conceive a parabolic curve to pass through the first three points B D F, which will very nearly coincide with the hyperbolic curve, when the ordinates are taken very near each other; and therefore, the area of the hyperbolic space A B F E will be very nearly equal to the parabolic space, their boundaries nearly coinciding. Let the abscissa be called x, and the ordinate y; then, from the nature of the parabola, we have $\frac{c^2}{t^2} \times (x^2 + t\,x) = y^2$, and $y = \frac{c}{t} \times \surd (x^2 + t\,x)$. By extracting the square root of $(x^2 + t\,x)$, which will be an infinite series, and multiplying by $\frac{c}{t}$, we shall have y. Let the infinite series, multiplied by $\frac{c}{t}$, be represented by $A + B\,x + C\,x^2 + D\,x^3$, &c., which is obviously a general expression for the ordinate y; and if we consider x to be composed of an infinite number of points, beginning with a cypher, the several values of it, as it increases from 0, may be represented by 0, 1, 2, 3, &c. Now, as all the ordinates (which are supposed to be extremely near

each other,) corresponding to the abscissas 0, 1, 2, 3, &c., make up the area of the space A B F E, let x be interpreted by 0, 1, 2, 3, &c., and we shall get

$$\begin{array}{ll}
A + B \times 0 + C \times 0^2 & = \text{A or A B the ordinate,} \\
A + B \times 1 + C \times 1^2 & \quad \text{when } x = 0. \\
A + B \times 2 + C \times 2^2 & \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
A + B \times \frac{x}{2} + C \times \frac{x^2}{4} & = \text{middle ordinate D C.} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
A + B \times x + C \times x^2 & = \text{third ordinate E F.}
\end{array}$$

The sum of all these is obviously the area of the space A B F E; that is, $A\,x + \frac{B\,x^2}{2} + \frac{C\,x^3}{3} = \text{area} =$

$$\frac{6\,A\,x + 3\,B\,x^2 + 2\,C\,x^3}{6} = \frac{x}{6} \times (6\,A + 3\,B\,x + 2\,C\,x^2)$$

From the middle equation, we have

$$\begin{array}{ll}
4\,A + 2\,B\,x + C\,x^2 = 4\,D\,C, & \text{and when } x = 0, \text{ we get} \\
A \qquad\qquad\qquad\quad = A\,B, & \text{when } x = A\,E, \text{ we get} \\
A + \quad B\,x + C\,x^2 = E\,F, & \text{add them together, and} \\
6\,A + 3\,B\,x + 2\,C\,x^2 = 4\,D\,C + A\,B + E\,F. & \text{But}
\end{array}$$

$(6\,A + 3\,B\,x + 2\,C\,x^2) \times \frac{x}{6} =$ area of the curve $\therefore$

$(A\,B + 4\,D\,C + E\,F) \times \frac{x}{6} =$ area of the curve, or

$(A\,B + 4\,D\,C + E\,F) \times \frac{D}{3}$; for $D = x \div 2$. Pursuing the same mode $(E\,F + 4\,G\,H + I\,K) \times \frac{D}{3} =$ area of the hyperbolic space E F K I; and $(I\,K + 4\,L\,M + N\,O) \times \frac{D}{3} =$ area of I K O N, &c., consequently the sum of all these areas will give the entire hyperbolic space A B O N

very nearly; that is, $(A\,B + 4\,C\,D + 2\,E\,F + 4\,G\,H + 2\,I\,K + 4\,L\,M + N\,O) \times \frac{D}{3} = (A + 4\,B + 2\,C \times \frac{D}{3}$. (Page 76.)

62. Let a = semi-transverse, b = semi-conjugate, x = the abscissa, y = the ordinate; then, $y^2 = \frac{b^2}{a^2} \cdot (2\,a\,x + x^2)$; hence, $x^2 + 2\,a\,x = \frac{a^2\,y^2}{b^2}$, and $x = \frac{a}{b} \times (b^2 + y^2)^{\frac{1}{2}} - a$, and $d\,x = \frac{a\,y\,d\,y}{b(b^2 + y^2)^{\frac{1}{2}}}$. And $d\,z =$

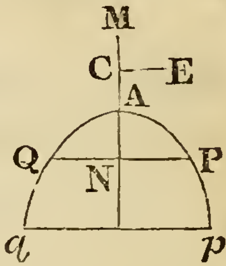
$$\left\{ d\,y^2 + \frac{a^2\,y^2\,d\,y^2}{b^2\,(b^2 + y^2)} \right\}^{\frac{1}{2}} = d\,y. \left\{ 1 + \frac{a^2\,y^2}{b^4 + b^2\,y^2} \right\}^{\frac{1}{2}} = d\,y \left\{ 1 + \frac{a^2\,y^2}{b^4} - \frac{a^2\,y^4}{b^6} + \frac{a^2\,y^6}{b^8} - \frac{a^2\,y^8}{b^{10}} + \text{\&c.} \right\}^{\frac{1}{2}} =$$

$$d\,y \left\{ 1 + \frac{a^2\,y^2}{2\,b^4} - \left(\frac{a^2}{2\,b^6} + \frac{a^4}{8\,b^8} \right) y^4 + \left(\frac{a^2}{2\,b^8} + \frac{a^4}{4\,b^{10}} + \frac{a^6}{16\,b^{12}} \right) y^6 - \text{\&c.} \right\} = d\,y \left\{ 1 + \frac{a^2\,y^2}{2\,b^4} - \frac{a^4 + 4\,a^2\,b^2}{8\,b^8} . y^4 + \frac{a^6 + 4\,a^4\,b^2 + 8\,a^2\,b^4}{16\,b^{12}} . y^6 - \text{\&c.} \right\} \therefore z = y \left\{ 1 + \frac{a^2}{6\,b^4} . y^2 - \frac{a^4 + 4\,a^2\,b^2}{40\,b^8} . y^4 + \frac{a^6 + 4\,a^4\,b^2 + 8\,a^2\,b^4}{112\,b^{12}} . y^6 - \text{\&c.} \right\} = \text{the arc A P.}$$

But the rule expressed algebraically is $\frac{15\,a\,b^2 + (19\,a^2 + 21\,b^2) \cdot x}{15\,a\,b^2 + (9\,a^2 + 21\,b^2) \cdot x} \times y$, which, if actually divided, the quotient will be found to differ but little from the preceding

series, which expresses the true value of the arcs; therefore, the rule approximates the truth. (Page 77.)

Note.—It is to be observed that $x = \frac{a}{b} \cdot (b^2 + y^2)^{\frac{1}{2}} - a.$



The length of an arc of an hyperbola may be found thus:—Conceive two curves to be drawn equi-distant from Q A P at an extremely small distance from it, as in the parabola; find the area of these two new figures taken as hyperbolas, from which they do not materially differ, by reason of their nearly coinciding with the hyperbola Q A P; then the difference between these two areas divided by the perpendicular distance between them will give the length of the curve Q A P nearly.

Hence, any curve may be rectified in this way, when its area can be found from certain dimensions.

63. Retaining the same notation, the equation of the hyperbola is $y = \frac{b}{a}(2\,a\,x + x^2)^{\frac{1}{2}}$. Hence $y\,d\,x = \frac{b}{a}(2\,a\,x + x^2)^{\frac{1}{2}}\,d\,x$. Now, in order to make the series expressing the area converge, let $w = \frac{x}{2\,a + x}$, from which $x = \frac{2\,a\,w}{1 - w}$ and $y\,d\,x = \frac{b}{a}\left\{\frac{4\,a^2\,w}{1 - w} + \frac{4\,a^2\,w^2}{(1-w)^2}\right\}^{\frac{1}{2}} \cdot \frac{2\,a\,d\,w}{(1 - w)^2} =$

$\frac{4\,a\,b\,w^{\frac{1}{2}}\,d\,w}{(1 - w)^3} = 2\,a\,b\,w^{\frac{1}{2}}\,d\,w\left\{1.2 + 2.3\,w + 3.4\,w^2 + 4.5\,w^3 \times \&c.\right\}$ by actual division; therefore, $\int y\,d\,x = 4\,a\,b\,w^{\frac{3}{2}}$

$\left\{\frac{1.2}{3} + \frac{2.3}{5} w + \frac{3.4}{7} w^2 + \&c.\right\}$ But as $w = \frac{x}{2a + x}$ is a proper fraction, the powers of w in this series converge, while the co-efficients diverge; therefore, in order to make the co-efficients, as well as the powers of w to converge, we multiply the series last obtained by $(1 - w$, also divide the factor $4\,a\,b\,w^{\frac{3}{2}}$ by the same quantity, and we get

$$f y\,d x = \frac{8\,a\,b\,w^{\frac{3}{2}}}{1 - w}. \left\{\frac{1.1}{1.3} + \frac{2.2}{3.5} w + \frac{3.3}{5.7} w^2 + \frac{4.4}{7.9} w^3 + \&c.\right\}$$

By repeating the operation the value of the series is not altered; hence

$$f y\,d x = \frac{8\,a\,b\,w^{\frac{3}{2}}}{(1 - w)^2}. \left\{\frac{1}{3} - \frac{1}{1.3.5}.w - \frac{1}{3.5.7} w^2 - \frac{1}{5.7.9}.w^3 - \&c.\right\}$$

but $y = \frac{b}{a}(2\,a\,x + x^2)^{\frac{1}{2}} = \frac{2\,b\,w^{\frac{1}{2}}}{1 - w}$, and $x = \frac{2\,a\,w}{1 - w} \therefore 2\,x\,y = \frac{8\,a\,b\,w^{\frac{3}{2}}}{(1 - w)^2}$; therefore,

$$f y\,d x = 2\,x\,y \left\{\frac{1}{3} - \frac{1}{1.3.5} w - \frac{1}{3.5.7} w^2 - \frac{1}{5.7.9} w^3 - \&c.\right\} = \text{area A P N.}$$

Hence area A P Q =

$$4\,x\,y\left\{\frac{1}{3} - \frac{1}{1.3.5} w - \frac{1}{3.5.7} w^2 - \&c.\right\} =$$

$$4\,x\,y\left\{\frac{1}{3} - \frac{1}{1.3.5}.\frac{x}{2\,a + x} - \frac{1}{3.5.7}.\frac{x^2}{(2\,a + x)^2} - \&c.\right\}$$

But $y = \frac{b}{a}(2\,a\,x + x^2)^{\frac{1}{2}}$. And the rule is

$\frac{21\,(2\,a\,x + \frac{5}{7}\,x^2)^{\frac{1}{2}} \times 4\,\sqrt{(a\,x)}}{75} \times \frac{4\,b\,x}{a}$, which being expanded, will produce a series nearly equal to the true expression found for the area. (See Fig. page 92, Appendix.) (Page 79.)

64. If we conceive a plane to pass through each of the lineal measuring units parallel to the ends; and the ends to be similarly divided by planes passing through each lineal measuring unit. parallel to the sides; then it is evident that the part cut off will be divided by the plane into as many cubes as there are squares in each end.

But the magnitude of the whole prism, or of any other of an equal base, is to the magnitude of the part whose height is the lineal measuring unit, as the height of the whole prism is to 1; therefore, the solidity of the whole is equal to that of the part repeated, as often as there are lineal measuring units in the height; that is, equal to the base multiplied by the height.

This rule is true for oblique prisms, as is evident by conceiving a right and an oblique prism, of equal bases and heights, to be made up of an infinite number of plates infinitely thin, all parallel to the base; when the prisms are of the same height, the right and oblique prisms will each require the same number of such plates, and therefore, they must be equal to each other, as they require the same number of equal plates to constitute them. (Page 83.)

65. It is proved in every work on solid Geometry, that every pyramid is one-third of a prism, having the same base and height; but the solidity of a prism is found by multiplying the area of the base by the height; therefore, the solidity of a pyramid is found by multiplying the area of the base by one-third of the height. (Page 87.)

66. In solid Geometry it is proved that every cone is the third part of a cylinder having the same base and altitude; but the solidity of a cylinder is found by multiplying the area of its base by its altitude; therefore the solidity of a cone is found by multiplying the area of its base by one-third of its altitude. (Page 88.)

67. Let A^2 and a^2 be equal to the areas A B and S D; P and p the perpendiculars from the vertex V, upon the planes of the bases A B and S D; and $h = P - p$, the height of the frustum. Put also $C =$ the entire solid. The content of the entire solid V A O B R is

$A^2 \times \frac{P}{3}$, and of the part V S Q D P $= a^2 \times \frac{p}{3}$, (Prob. VI.)

and the difference between these two solids is the content of the frustum; that is,

$$A^2 \times \frac{P}{3} - a^2 \times \frac{p}{3} = A^2 \times \frac{h}{3} + (A^2 - a^2) \times \frac{p}{3};\text{ but}$$

$$A^2 : a^2 :: A R^2 : S P^2 \text{ (20. VI.); hence,}$$

$$A : a :: A R : S P \text{ (22. VI.); but}$$

A R : S P :: R V : P V :: P : p (4. VI.); therefore, by equality of ratios, A : a :: P : p; and then, by division A — a : a :: h : p; but

$$A - a : a :: A^2 - a^2 : a \times (A + a);\text{ hence}$$

$$A^2 - a^2 : A a + a^2 :: h : p,\text{ and}$$

$$(A^2 - a^2) \times \frac{p}{3} = (A a + a^2) \times \frac{h}{3};\text{ therefore,}$$

$(A + A\,a + a^2) \times \frac{h}{3} =$ the content of the frustum, which is the rule. (Page 89.)

68. The Demonstration given of the last rule applies to this. (Page 90.)

69. Let the planes A O C G, A R B O, B R G C, be trapezoids, the latter being any how inclined to the former, and C G, A O, B R, parallel to one another; let also the wedge be divided into two pyramids by the plane B O G; then, B is the vertex of the pyramid, whose base is the triangle O G C; and G is the vertex of the pyramid which has A R B O for its base. Again, let S D be perpendicular to A O and R B, and D T perpendicular to A O and G C; then, the triangular plane S D T is perpendicular to the three parallel edges B R, A O, C G. Put $p =$ the perpendicular of the triangle, S D T, let fall from the vertex S, upon the base D T, (produced if necessary,) and P the perpendicular let fall from T upon S D, (produced if necessary); hence P and p are the heights of the pyramids A R B O G and G C O B respectively.

Now, $\frac{C\,G \times D\,T}{2} =$ the area of the triangle O G C;

and $\frac{G\,C \times T\,D}{2} \times \frac{p}{3} = \frac{G\,C}{3} \times \frac{D\,T \times p}{2} =$ the solid

content of the pyramid O G C B, and $\frac{A\,O + R\,B}{2} \times S\,D$

$=$ the area of the trapezoid A R B O; and $\frac{A\,O + R\,B}{2} \times$

$S\,D \times \frac{P}{3} = \frac{A\,O + R\,B}{3} \times \frac{S\,D \times P}{2} =$ the solid con-

tent of the other pyramid. But the area of the triangular

section S D T $= \dfrac{DT \times p}{2} = \dfrac{SD \times P}{2}$; therefore

sum $\dfrac{GC}{3} \times \dfrac{DT \times p}{2} + \dfrac{AO + RB}{3} \times \dfrac{SD \times P}{2}$ or the

content of the wedge is $\left(\dfrac{GC}{3} + \dfrac{AO + RB}{3}\right)$ multiplied

by $\dfrac{DT \times p}{2}$ or by $\dfrac{SD \times P}{2}$; that is, one-third of the

sum of the parallel edges, multiplied by the area of the triangle S D T, which is the rule. (Page 91.)

70. Dr. Hutton gives the following demonstration of this rule :—

Since it is evident that, according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism, of the same height and breadth with the wedge, and length equal to that of the edge, by a pyramid of the same height and breadth at the base also, and the length of whose base is equal to the difference of the length of the edge and base of the wedge; we shall have the content $\frac{1}{2}\,b\,l\,h \pm \frac{1}{3}\,b\,h \times (\pm L \mp l) = \frac{1}{2}\,b\,l\,h + \frac{1}{3}\,b\,h \times (L - l) = \frac{1}{6}\,b\,h \times (3\,l + 2\,L - 2\,l) = \frac{1}{6}\,b\,h \times (2\,L + l)$.

Cor. If $l = L$, the rule will become $\frac{1}{6}\,b\,h \times 3\,L = \frac{1}{2}\,b\,h\,L = \frac{1}{2}$ a prism of the same base and height, as it ought.

Scholium. It is evident that, whether the two ends, or the two sides of the wedge be equally or unequally inclined to the base, it will make no difference in the rule. (Page 91.)

71. Conceive the frustum to be cut by a plane passing through the opposite edges A O and P D, which will evidently divide it into two wedges A R B O D P and P S Q D O A. Let B I represent the perpendicular distance between the ends A R B O and S P D Q; also let B O represent the distance of A O from R B, and D Q the perpendicular distance of

P D from S Q; then it is obvious that $\frac{\text{B I} \times \text{B O}}{2}$; and $\frac{\text{B I} \times \text{D Q}}{2}$ will be the respective areas of the two triangular sections of the two wedges, which are perpendicular to the edges B R, A O, S Q, and P D. Then, by the last rule, the content of the wedge A R B O D F will be

$\frac{\text{R B} + \text{A O} + \text{P D}}{3} \times \frac{\text{B I} \times \text{B O}}{2}$; and the content of the wedge P S Q D O A will be $\frac{\text{S Q} + \text{A O} + \text{P D}}{3} \times \frac{\text{B I} \times \text{D Q}}{2}$, and the sum of both will be the solidity of the prismoid. (Page 92.)

72. It has been shown before that the prismoid is composed of two wedges, whose bases are the two ends of the prismoid, and whose heights are each equal to that of the prismoid; therefore, by the last Problem, Rule II., its solidity is $= [(2\,\text{L} + l)\,\text{B} + (2\,l + \text{L})\,b] \times \frac{1}{6}\,h$; and as $\frac{1}{2}\,\text{L} + \frac{1}{2}\,l = \text{M}$, and $\frac{1}{2}\,\text{B} + \frac{1}{2}\,b = m$, are length and breadth of a section parallel to, and equally distant from, each end, we shall get

$$[(2\,\text{L} + l)\,\text{B} + (2\,l + \text{L})\,b] \times \tfrac{1}{6}\,h, \text{ or}$$

$(2\,\text{B}\,\text{L} + \text{B}\,l + 2\,b\,l + b\,\text{L}) \times \frac{1}{6}\,h = (\text{B}\,\text{L} + b\,l + 4\,\text{M}\,m) \times \frac{1}{6}\,h$; that is, the sum of the areas of the two ends, and 4 times the section in the middle, multiplied by $\frac{1}{6}\,h$.

As every prismoid and cylindroid may be conceived to consist of an infinite number of rectangular pyramids, it is evident that the last rule will answer for any prismoid or cylindroid, of whatever figure the opposite ends may be. (Page 93.)

73. This rule may be easily deduced from the preceding ones. Other rules are given, which find the true solidity only when the middle section, between the two ends, is similar to the two ends, which never can be, except when the parallel ends are similar ellipses; that is, the transverse and conjugate diameters at each end parallel to each other; this can never happen but when the solid is the frustum of an elliptical cone. (Page 93.)

74. The reason of the first rule is evident from Solid Geometry, where it is demonstrated that a sphere is $\frac{2}{3}$ of its circumscribed cylinder. But the diameter of the base of the cylinder, and its altitude, are each equal to the diameter of the sphere; therefore (d being the diameter), $d^2 \times \cdot 7854$ is the area of the base, which being multiplied by the height of the cylinder, will give its solidity; that is, $d^2 \times \cdot 7854 \times d = d^3 \times \cdot 7854 =$ the content of the cylinder, the two-thirds of which will be the solidity of the sphere; that is, $d^3 \times \cdot 7854 \times \frac{2}{3} = d^3 \times \cdot 5236 =$ the content of the sphere, which is the first rule.

The reason of the second rule is equally obvious. For the surface of a sphere is equal to the circumference of one of its great circles multiplied by its diameter; that is, $c \times d$, c being the circumference, and d the diameter of the sphere. Now, the sphere may be considered as made up of an infinite number of pyramids, whose bases compose the surface of the sphere, and all the vertices meeting in the centre, their common altitude or height being equal to the radius of the sphere, or half the diameter; and therefore, its solid content, or the solid content of any spherical pyramid, being a part contained within right lines drawn from the surface to the centre, is equal to a pyramid whose base is equal to the spherical surface, and height equal to the radius, or half the diameter; that is, $c \times d \times \frac{d}{6}$, which is the rule. (Page 94.)

75. Let A C n be a triangular pyramid, whose base n A is infinitely small; the sphere, as was said before, may be conceived to be composed of an infinite number of such pyramids, whose bases constitute the surface of the sphere, their altitudes being the radius of the sphere, and the centre their common vertex; then, by the last rule, the solidity of the sphere, or any sector thereof, is equal to a pyramid, the area of whose base is the spherical surface, and its altitude the radius of the sphere.

Let O D (h) be the height of the segment A D B; then, $d \times 3{\cdot}1416 \times h =$ the spherical surface, and $d \times 3{\cdot}1416 \times h \times \frac{1}{3}$ C D $= 3{\cdot}1416 \times d\,h \times \frac{1}{6}\,d = {\cdot}5236\,d^2\,h =$ the solidity of the sector A C B D. But E O $\times$ O D $=$ A O^2; that is, $(d - h) \times h =$ A O^2, and A O$^2 \times 3{\cdot}1416 \times \frac{1}{3}$ O C $=$ the solidity of the cone A C B $= (d - h)\,h \times$

$$3{\cdot}1416 \times \tfrac{1}{3}\left(\frac{d}{2} - h\right) = (d - h) \times h \times {\cdot}5236 \times (d - 2\,h)$$

$= (d^2\,h - 3\,d\,h^2 + 2\,h^3) \times {\cdot}5236$, which, taken from the solidity of the sector, leaves the solidity of the segment; that is, $({\cdot}5236\,d^2\,h) - d^2\,h - 3\,d\,h^2 + 2\,h^3) \times {\cdot}5236 = (3\,d\,h^2 - 2\,h^3) \times {\cdot}5236 = (3\,d - 2\,h) \times h^2 \times {\cdot}5236$, which is the first rule. This rule will hold true when h is less than $\frac{1}{2}\,d$.

Let $r =$ A O, the radius of the segment's base; then,

$$(d - h) \times h = r^2\text{; hence, } d = \frac{r^2 + h^2}{h}\text{; then substitute}$$

for d, and the rule becomes $\left(\dfrac{3\,r^2 + 3\,h^2}{h} - 2\,h\right) \times h^2 \times$

${\cdot}5236 = (3\,r^2 + h^2) \times h \times {\cdot}5236$, which is the second rule, and is always to be used when the radius of the sphere is not given. (Pages 95, 96.)

76. Put H $=$ height of the greater segment, and $h =$ height of the less; R $=$ radius of the greater base, and r

$=$ radius of the less. Then, it is obvious that the difference between these two segments will be the zone required; that is, $(3\,R^2 + H^2) \times H \times \cdot 5236 - (3\,r^2 + h^2) \times h^2 \times \cdot 5236 = [(3\,R^2\,H + H^3) - (3\,r^2\,h + h^3)] \times \cdot 5236$. Put $d =$ the diameter of the sphere, and then, from the property of the circle, we get $(d - H) \times H = R^2$, and $(d - h) \times h = r^2$.

Hence, $d = \dfrac{R^2 + H^2}{H}$, and $d = \dfrac{r^2 + h^2}{h}$; consequently, $\dfrac{R^2 + H^2}{H} = \dfrac{r^2 + h^2}{h}$, and putting $a = H - h$, we get $[(3\,R^2\,H + H^3) - 3\,r^2\,h + h^3)] \times \cdot 5236\ (R^2 + r^2 + \frac{1}{3}\,a^2) \times a \times 1\cdot 5708$. Now, if one of the radii pass through the centre, we get $R^2 = \dfrac{d^2}{4} = C\,O^2 + G\,O^2 = r^2 + a^2$; hence, the expression becomes $(r^2 + \frac{2}{3}\,a) \times a \times 3\cdot 1416 = (\frac{1}{4}\,d^2 - \frac{1}{3}\,a^2) \times a \times 3\cdot 1416$. Hence, $(r^2 + \frac{2}{3}\,a^2) \times a \times 6\cdot 2832 = (\frac{1}{4}\,d^2 - \frac{1}{3}\,a^2) \times a \times 6\cdot 2832$ expresses the solidity of the middle zone A B D C, being double the former, where a is $\frac{1}{2}$ the altitude, and $r =$ half the diameter of each end. Put $A =$ the whole altitude, and $d' = 2\,r$, the diameter of each end; and the expression becomes $(d'^2 + \frac{2}{3}\,A^2) \times A \times \cdot 7854 = (d^2 - \frac{1}{3}\,A^2) \times A \times \cdot 7854$. (Page 97.)

77. Put $F\,C = a$, $F\,S = c$, and $r =$ radius; conceive an infinite number of ordinates, y, y', y'', &c., to be drawn as in the figure, and let the distance between every two of them be represented by x. Then, $O\,V^2 = x^2$; therefore $r^2 - x^2 = (c + y')^2 = c^2 + 2\,c\,y' + y'^2$; hence, $r^2 - c^2 -$

$2\,c\,y' - x^2 = y'^2$; but $S\,C^2\,(r^2) - F\,S^2\,(c^2) = F\,C^2\,(a^2)$;

therefore, $a^2 - 2\,c\,y' - x^2 = y'^2$. Now, if we take $x = 0$, 1, 2, 3, &c., we get

$$a^2 - 2\,c\,y = y^2$$

$$a^2 - 2\,c\,y' - 1^2 = y'^2$$

$$a^2 - 2\,c\,y'' - 2^2 = y''^2$$

$$a^2 - 2\,c\,y''' - 3^2 = y'''^2$$

&c. &c. &c. &c.

$$a^2 - 2c \times 0 - a^2 = 0^2$$

But if we conceive the spindle to revolve about the chord A C, the sum of all the circles whose radii are y, y', y'', &c. will be the solidity of A B L, and the area of these circles is $\left\{(2y)^2[= 4\,y^2] + (2\,y')^2\,[= 4\,y'^2] + \text{\&c.}\right\} \times {\cdot}7854 =$ $(y^2 + y'^2 + y''^2 + \text{\&c}) \times 4 \times {\cdot}7854 = y^2 + y'^2 + y''^2$ $+ \text{\&c.})\ 3{\cdot}1416$. But $y^2 + y'^2 + y''^2 + \text{\&c.}) =$ the sum of the left-hand members of the equations; therefore, the sum of the left hand members multiplied by 3·1416 will give the solidity of the part A B L. The sum $a^2 + a^2$, &c. $= a^3$, sum of $2\,c\,y + 2\,c\,y' + 2\,c\,y'' + \text{\&c.} = 2\,c \times$ A B F. also, $0^2 + 1^2 + 2^2 + 3^2 + \ldots\ldots\ a^2 = \frac{a^3}{3}$; therefore, the sum of the left hand members of the equations is $a^3 - 2\,c$ $\times$ space A B F $- \frac{a^3}{3} = \frac{2\,a^3}{3} - 2\,c \times$ space A B F $= \Big(\frac{a^3}{3}$

$-c \times$ space A B F$\Big) \times 2$. Then, $\left(\frac{a^3}{3} - c \times \text{space}\right.$ $\left.\text{A B F}\right) \times 2 \times 3{\cdot}1416 =$ the solidity of A B L, which is half the spindle, therefore, the whole spindle will be $\left(\frac{a^3}{3} - c\right.$ $\left.\times \text{ space A B F}\right) 4 \times 3{\cdot}1416 = \left(\frac{a^3}{3} - c \times \text{space A B F}\right)$ $\times$ 12·5664, which is the rule. (Page 98.)

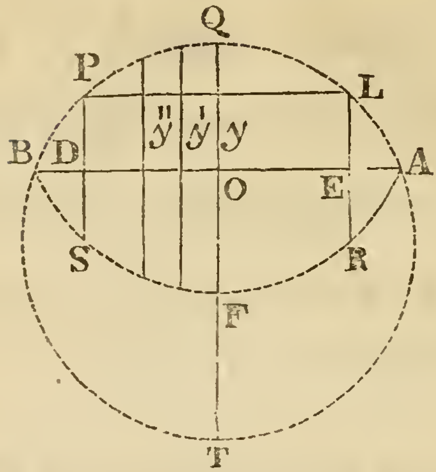
78. By the last, we have $r^2 - x^2 = (c + y')^2 = c^2 + 2\,c\,y' + y'^2$, and $r^2 - c^2 - 2\,c\,y' - x^2 = y^2$; put $r^2 - c^2 = a^2$; then,

$$a^2 - 2\,c\,y - x^2 = y'^2$$

Put 0, 1, 2, 3, &c. for x, then,

$$a^2 - 2\,c\,y - 0^2 = y^2$$
$$a^2 - 2\,c\,y' - 1^2 = y'^2$$
$$a^2 - 2\,c\,y'' - 2^2 = y''^2$$
$$a^2 - 2\,c\,y''' - 3^2 = y'''^2$$
&c. &c. &c. &c.
$$a^2 - 2\,c\,\text{D P} - \text{D O}^2 = y^2$$

From what has been said in the last Demonstration, we have $a^2 \times \text{D O} - 2\,c \times \text{space D P Q O} - \frac{\text{D O}^3}{3} =$ sum of the left-hand members of the equation. But



$$a^2 \times DO - 2c \times \text{space } DPQO - \frac{DO^3}{3} =$$

$$(a^2 - DO^2) \times DO - 2c \times \text{space } DPQO =$$

$$\left(a^2 - \frac{DO^2}{3}\right) \times DO - c \times \text{space } PDELQ.$$

Then this multiplied by $2 \times 3{\cdot}1416$ will, from what has been said in the last, give the solidity of the frustum; that is, $(a^2 - \frac{DO^2}{3} \times DO - c \times \text{space } PDELQ \times 6{\cdot}2832$, which is the rule. (Page 99.)

79. It was shown in Proposition XIV. Cor. 2. Ellipse, that the solidity of the spheroid is $\frac{2\,c^2\,t\,n}{3}$, when t is the transverse, c the conjugate, and $n = {\cdot}7854$; but $\frac{2\,c^2\,t\,n}{3}$ $= \frac{2}{3}\,n \times c^2\,t = \frac{2}{3} \times {\cdot}7854 \times c^2\,t = {\cdot}5236 \times c^2\;t$, which is the rule.

The solidity of the oblate spheroid is $\frac{2\,n\,t^2\,c}{3} = \frac{2}{3}\,n \times$

$t^2 c = \cdot 5236 \times t^2 c$.—See Prop. XV. Cor. 1. Ellipse. (Page 100.)

80. Let the segment be that of a prolate spheroid; then it is evident that any plane section of the inscribed sphere, parallel to the transverse or fixed diameter, is to the corresponding section of the spheroid, as the conjugate to the transverse; therefore, the sum of all the circles which form the segment of the inscribed sphere, is to the sum of all the ellipses which form the corresponding segment of the spheroid, as the conjugate to the transverse; but the content of the segment of the inscribed sphere is (Demonstration 75,) $(3c - 2h)h^2 \times \cdot 5236 \therefore \frac{t}{c}(3c - 2h)\ h^2 \times \cdot 5236 =$ the content of the segment parallel to the fixed or perpendicular to the revolving axis. In a similar manner the rule may be proved when the segment is that of an oblate spheroid.

81. Let $x =$ the height of the segment which is an abscissa of the revolving ellipse, let y be the corresponding ordinate, and t and c the transverse and conjugate diameters, then $y^2 = \frac{c^2}{t^2} \times (t\,x - x^2)$; hence the sum of all the circles constituting the volume of the segment, will (Propositions II. and III. Arithmetic of Infinites,) be $\frac{4\,n\,c^2}{t^2}(\frac{1}{2}\ t\ x^2 - \frac{1}{3}\ x^3)$ $= \frac{c^2}{t^2} \times \frac{2}{3} n \times [(3\,t - 2\,x) \times x^2] = \frac{c^2}{t^2} \times \cdot 5236 \times [(3\,t - 2\,x) \times x^2]$, which is the rule. (Page 101.)

82. Let $f =$ A B the fixed axis, $r =$ E F the revolving axis, } For the prolate spheroid.
$h = v\,o$ the height of the middle frustum.
D $=$ K I the diameter of one end of the spherical zone.
$d = m\,n$ the corresponding diameter of the spheroidal zone.

Then, by the Demonstration 76, the solidity of the middle zone of the sphere is $(f^2 - \frac{1}{3} h^2) h \times \cdot 7854 = (3 f^2 - h^2) \times h \times \cdot 2618$; but $f^2 : r^2 :: (3 f^2 - h^2) h \times \cdot 2618$: the solidity of the spheroidal zone, by Prop. XIV. Cor 3. Ellipsis; and $C D^2 : E F^2 :: I K^2 : m n^2$; that is, $f^2 : r^2 :: f^2 - h^2 : d^2$; hence, $f^2 = \frac{r^2 h^2}{r^2 - d^2}$; therefore, $\frac{r^2 h^2}{r^2 - d^2} : r^2 :: \left(\frac{3 r^2 h^2}{r^2 - d^2} - h^2\right) \times h \times \cdot 2618$: the solidity of the spheroidal zone $= (2 r^2 + d^2) h \times \cdot 2618$.

Putting $f = E F$, and $r = A B$, a similar result will be obtained; for the content of the middle zone of an oblate spheroid. (Page 102.)

83. It was proved in Prop. VIII. *Arith. of Infinites*, that the parabolic conoid is half of a cylinder of the same base and height. But the solidity of the cylinder is $D^2 \times \cdot 7854 \times h$, ($h$ being the height of the cylinder); therefore, the solidity of the conoid is $\frac{D^2 \times \cdot 7854 \times h}{2} = D^2 \times \cdot 3927 \times h$, which is the rule. (Page 103.)

84. It was proved in Prob. VIII. Cor. *Arith. of Infinites*, that the solidity of the lower frustum of a parabolic conoid is equal to half the sum of both bases multiplied by the height of the frustum. If D be the diameter of the greater base, and d the diameter of the less, their areas are $D^2 \times \cdot 7854$, and $d^2 \times \cdot 7854$; then, their sum is $(D^2 + d^2 \times \frac{\cdot 7854}{2} = (D^2 + d^2) \times \cdot 3927$, which multiplied by the height of the frustum will give its solidity, viz. $(D^2 + d^2) \times \cdot 3927 \times h$, is the solidity, which is the rule. (Page 103.)

85. It was shown in Prop. IX. *Arith. of Infinites*, that every parabolic spindle is equal to $\frac{8}{15}$ of its circumscribed cylinder. But the contents of the circumscribed cylinder is $D^2 \times \cdot 7854 \times l$, D being the middle diameter, and l the length; therefore $\frac{8}{15} \times D^2 \times \cdot 7854 \times l$, is the solidity of the spindle, which is the rule. (Page 104.)

86. In Prop. IX. Cor. *Arith. of Infinites*, the equation for the solidity of the frustum is $2\ D^2 + C^2 - \frac{4}{10}\ d^2) \times L \times \cdot 2618$, where D = middle diameter, C = diameter of the end, d = difference of diameters, and L = the length, which is the rule. (Page 105.)

87. Put $A\ m = R$, and $x = V\ m$ the height; then, $R^2 \times 3 \cdot 1416$ = area of the base, and $R^2 \times x \times 3 \cdot 1416$ = solidity of the cylinder of the same base and height as the hyperbolic conoid is to the cylinder of the same base and height as $\frac{1}{2}\ t + \frac{1}{3}\ x$ to $t + x$; therefore, $t + x : \frac{1}{2}\ t + \frac{1}{3}\ x :: R^2 \times x \times 3 \cdot 1416$: hyperbolic conoid, or $t + x$: $\frac{3\ t + 2\ x}{6} :: R^2 \times x \times 3 \cdot 1416$: hyperbolic conoid, which is, therefore, equal $\frac{(3\ t + 2\ x)}{(t + x)} \times R^2 \times x \times \frac{3 \cdot 1416}{6} = \frac{(3\ t + 2\ x) \times R^2 \times x \times \cdot 5236}{t + x}$, which is the rule, t being the transverse. (Page 106.)

88. This rule is proved in Demonstration 72, where it is shown that four times the area in the middle, added to the areas of the two ends, and the sum multiplied by $\frac{1}{6}$ of the height, gives the solidity, but $\cdot 1309$

being $\cdot 7854 \div 6$, then, $(4\,D^2 + d^2 + d'^2) \times h \times \frac{\cdot 7854}{6} =$ $(4\,D^2 + d^2 + d'^2) \times h \times \cdot 1309$, which is the rule. (Page 107.)

89. This rule is the same as the last, and the Demonstration is the same.

90. Let *m o n v* be a cylindrical ring, the diameter of a section of which is A C, and the mean length *m o n v* passing through its centre. Then $A\,C + C\,D = m\,n$; therefore, the mean length of the cylinder is $m\,n \times 3\cdot 1416$; and the area of a section A C is $A\,C^2 \times \cdot 7854$; but the solidity of a cylinder is found by multiplying the area of its base, which is here $A\,C^2 \times \cdot 7854$, by its length, which is $m\,n \times 3\cdot 1416$; that is, $A\,C^2 \times \cdot 7854 \times m\,n \times 3\cdot 1416 = A\,C^2 \times (A\,C + C\,D) \times 2\cdot 4674$, which is the rule. (Page 108.)

91. Let A B C be the tetraedron; from C let fall the perpendicular C E, on the opposite side A B D, and join E A. Then $A\,C^2 = A\,E^2 + E\,C^2$; but $\frac{1}{3}\,A\,C^2 = \frac{1}{3}\,A\,B^2 = A\,E^2$; therefore, $\frac{2}{3}\,A\,C^2 = E\,C^2$. Hence, $A\,C\,\sqrt{\frac{2}{3}} = E\,C$; but $A\,B\,D = \frac{1}{4}\,A\,B^2\,\sqrt{3} = \frac{1}{4}\,A\,C^2\,\sqrt{3}$; then the solidity will be equal to the area of the base multiplied by $\frac{1}{3}$ of the altitude; that is, $\frac{1}{3}\,C\,E \times A\,B\,D = \frac{1}{3}\,A\,C\,\sqrt{\frac{2}{3}} \times \frac{1}{4}\,A\,C^2\,\sqrt{3} = \frac{1}{12}\,A\,C^3\,\sqrt{2}$, the solidity, which is the rule.

The reason of the second rule is obvious, from a property in Solid Geometry, viz., that similar solids are to one another as the cubes of their like sides; and the tabular numbers being the content of solids whose sides are 1; therefore, the cube of any side multiplied by the tabular number corresponding to the figure, will give its solidity. (Page 111.)

92. Let E be the centre of the solid, or the middle of the diagonal A C, join D E, which is equal to A E.

The solid is evidently composed of two equal square pyramids, the common base of which, A B C F, is equal to the square of the lineal side of the solid, the altitude of each being equal to D E, or A E, half the diagonal of that square. Now, $A B^2 = A B C F$; but the area $A B C F \times \frac{2}{3} A E = A B^2 \times \frac{1}{3} A C = \frac{1}{3} A B^2 \sqrt{(A B^2 + B C^2)} = \frac{1}{3} A B^2 \sqrt{2 A B^2}) = \frac{1}{3} A B^3 \sqrt{2}$, which is the rule. (Page 112.)

93. Let $\triangle$ represent a solid angle of the dodecaedron, and connect the extremities of the sides A B, A C, A D, of the faces which form the angle, by the lines B C, C D, and D B, forming an equilateral triangle B C D, within the solid, on the centre of which, let fall the perpendicular A E; join the centre F, of one of the faces, and the points A and C.

The angle C A D contains 108 degrees, the sine of which is $\frac{1}{4} \sqrt{(10 + 2 \sqrt{5})}$ to the radius 1.

The angle A D C contains 36 degrees, the sine of which is $\frac{1}{4} \sqrt{(10 - 2 \sqrt{5})}$. (*Trigonometry.*)

Hence, $\sqrt{(10 - 2 \sqrt{5})} : \sqrt{(10 + 2 \sqrt{5})} :: A C : D C =$

$$A C \sqrt{\frac{10 + 2 \sqrt{5}}{10 - 2 \sqrt{5}}} = A C \sqrt{\frac{5 + \sqrt{5}}{5 - \sqrt{5}}} = A C \sqrt{\frac{(5 + \sqrt{5})^2}{25 - 5}} = A C \times \frac{5 + \sqrt{5}}{2 \sqrt{5}} = \frac{1 + \sqrt{5}}{2} \times A C.$$

In somewhat a similar manner, we find $C E = \frac{1}{3} C D \sqrt{3} = C D \sqrt{\frac{1}{3}} = \frac{1 + \sqrt{5}}{2 \sqrt{3}} \times A C$; hence, $E A =$

$$\sqrt{(A C^2 - C E^2)} = \sqrt{\left\{ A C^2 - \left(\frac{1 + \sqrt{5}}{2 \sqrt{3}}\right)^2 A C^2 \right\}}$$

$$= A\,C \sqrt{\left(1 - \frac{3 + \sqrt{5}}{6}\right)} = A\,C \sqrt{\frac{3 - \sqrt{5}}{6}}$$

Now, the chord of an arc being a mean proportional between its versed sine and the diameter; A E being the versed sine whose chord is A C, and its diameter equal to that of the circumscribed sphere; we have

$$A\,C^2 \div 2\,A\,E = A\,C^2 \div 2\,A\,C \sqrt{\frac{3 - \sqrt{5}}{6}} = \tfrac{1}{2}\,A\,C$$

$$\sqrt{\frac{6}{3 - \sqrt{5}}} = \tfrac{1}{2}\,A\,C \sqrt{6 \times \frac{3 + \sqrt{5}}{9 - 5}} = \tfrac{1}{2}\,A\,C \times \sqrt{}$$

$$\left(3 \times \frac{6 + 2\sqrt{5}}{4}\right) = A\,C \times \frac{1 + \sqrt{5}}{4} \times \sqrt{3} =$$

$$\frac{\sqrt{3} + \sqrt{15}}{4}\,A\,C = R,$$ the radius of the circumscribed sphere.

Again, the angle A F C contains 72 degrees, whose sine is $\frac{1}{4}\sqrt{(10 + 2\sqrt{5})}$. The angle A C F is 54 degrees, whose sine is $\frac{1 + \sqrt{5}}{4}$. Hence, $\sqrt{(10 + 2\sqrt{5})} : 1 + \sqrt{5} :: A\,C : A\,F = \frac{1 + \sqrt{5}}{\sqrt{(10 + 2\sqrt{5})}} \times A\,C = A\,C \sqrt{\frac{5 + \sqrt{5}}{10}}$. Now, it is obvious that the radius of the circumscribed sphere is the hypothenuse of a right-angled triangle, whose legs are A F and the radius of the inscribed sphere; hence, we have $\sqrt{(R^2 - A\,F^2)} = \sqrt{}$

$$\left[\left(\frac{\sqrt{3} + \sqrt{15}}{4}\,A\,C\right)^2 - \frac{5 + \sqrt{5}}{10}\,A\,C^2\right] = A\,C \sqrt{}$$

$$\left(\frac{18 + 6\sqrt{5}}{16} - \frac{5 + \sqrt{5}}{10}\right) = A\,C \sqrt{\frac{25 + 11\sqrt{5}}{40}}$$

$= r$, the radius of the inscribed sphere.

Now, the solidity of any regular solid is equal to the surface multiplied by $\frac{1}{3}$ of the radius of the inscribed sphere, and the surface, as will be shown hereafter, is equal to 15 $A^2 \sqrt{\frac{5 + 2\sqrt{5}}{5}}$; therefore, $B \times \frac{1}{3} r = 15 A^2 \sqrt{\frac{5 + 2\sqrt{5}}{5}} \times \frac{1}{3} A \sqrt{\frac{25 + 11\sqrt{5}}{40}} = 5 A^3 \times \frac{47 + 21\sqrt{5}}{40}$ = C the solidity, A being the lineal side, B the surface, and C the solidity. (Page 12.)

94. Let A be the solid angle of the icosaedron, formed by 5 triangles, whose bases form the pentagon B C D E F, on the centre of which let fall the perpendicular A C, join B G.

In one of the steps of the last demonstration, it was shown that $B G = A B \sqrt{\frac{5 + \sqrt{5}}{10}}$, and the radius of the circle circumscribing one of the faces A B C, of the solid, $= A \sqrt{\frac{1}{3}}$. But the radius of the circumscribing sphere is $R = \frac{A B^2}{2 A G} = \frac{A B^2}{2 \sqrt{(A B^2 - B G^2)}} =$

$$\frac{A B^2}{2 \sqrt{\left(A B^2 - \left(\frac{5 + \sqrt{5}}{10}\right) A B^2\right)}} = \frac{A B}{2 \sqrt{\left(1 - \frac{5 + \sqrt{5}}{10}\right)}}$$

$$= \frac{A B}{2 \sqrt{\frac{5 - \sqrt{5}}{10}}} \sqrt{\ } \tfrac{1}{2} A B \sqrt{\frac{10}{5 - \sqrt{5}}} = A B \sqrt{\ }$$

$$\left(\frac{10}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}\right) = \tfrac{1}{2}\,A\,B \sqrt{\frac{10 \times (5+\sqrt{5})}{25-5}}$$
$$= A\,B \sqrt{\frac{5+\sqrt{5}}{8}}$$

Now, R is the hypothenuse of a right-angled triangle, of which the one leg is Q ($= A\,B\sqrt{\tfrac{1}{3}}$) the radius of the circle circumscribing one of the faces A B C, and the other the radius r of the inscribed sphere.

Hence $r = \sqrt{(R^2 - Q^2)} = \sqrt{\left(\frac{5+\sqrt{5}}{8}\,A\,B^2 - \frac{1}{3}\,A\,B^2\right)} = A\,B\sqrt{\frac{15+3\sqrt{5}-8}{8\times 3}} = A\,B\sqrt{\frac{7+3\sqrt{5}}{24}}$

If the whole surface be denoted by B, and the solidity by S, we shall have

$S = \frac{1}{3}\,r\,B = \frac{1}{3}\,A\,B\sqrt{\frac{7+3\sqrt{5}}{24}}\;5\,A\,B\sqrt{3} = \frac{5}{3}\,A\,B^3\sqrt{\frac{7+3\sqrt{5}}{8}} = \frac{5}{6}\,A\,B^3\sqrt{\frac{7+3\sqrt{5}}{2}}$, which is the rule. (Page 113.)

95. The area of an equilateral triangle (Problem VI. Section II.) is $\frac{A^2}{4}\sqrt{3}$, A being one of the sides; then, the area of the four faces will be $A^2\sqrt{3}$, which is the first rule. The reason of the second rule is obvious from the property, that similar surfaces are to each other as the squares of their like sides. (Page 115.)

96. The hexaedron is composed of six square faces, the area of each being A^2, (A being the side,) therefore, $6\,A^2$ is the whole surface. (Page 115.)

97. By Problem VI. Section II. the area of one of the faces is $\frac{A^2}{4}\sqrt{3}$, (A being a side,) therefore, the surface of the 8 faces of the octaedron is $\frac{A^2}{4}\sqrt{3} \times 8 = 2A^2\sqrt{3}$, which is the rule. (Page 116.)

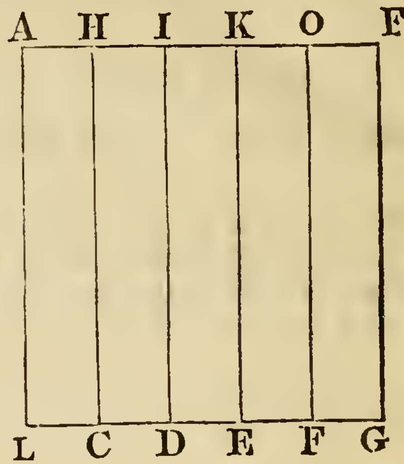
98. In Table II. the area of the pentagon, whose side is 1, is $\frac{5}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})}$ which being multiplied by 12, will give the whole surface, that is, $12 \times \frac{5}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})} = 15\sqrt{(1+\frac{2}{5}\sqrt{5})}$, which is the rule. (Page 116.)

99. By Problem VI. Section II. the area of one of the faces is $\frac{A^2}{4}\sqrt{3}$ (A being one of the sides); but the figure has 20 such faces; therefore, $20 \times \frac{A^2}{4}\sqrt{3} = 5A^2\sqrt{3}$ is the surface of the whole solid. (Page 117.)

Table IV. may be calculated from Table II. by multiplying the tabular numbers there, corresponding to the faces of the regular bodies, by the number of such faces forming the solid. Thus, 4 times the tabular number corresponding to an equilateral triangle will be the tabular number corresponding to the tetraedron; 6 times the tabular number answering to a square will be the tabular number answering to the hexaedron; 8 times the tabular number answering to the triangle will be the tabular number that answers the octaedron; and so of the rest. (Page 117.)

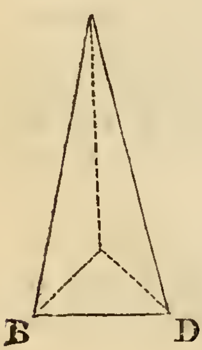
100. If we conceive the pentagonal prism C D E, &c., to be formed of pasteboard, the upright surface of it will, if

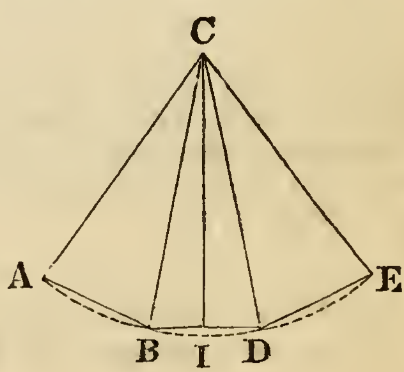
unfolded, form a parallelogram A F G L, whose altitude is equal to that of the prism, and base L G equal to the circum-



ference or perimeter of the pentagon; but the area of the parallelogram is A L $\times$ L G; therefore, the convex surface of the prism is $h \times p$, h being its height, and p ($=$ L G) its circumference or perimeter, to which the areas of both ends are to be added to find the surface of the entire prism, which is the rule. (Page 119.)

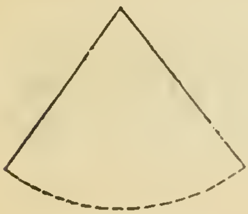
101. If we conceive a triangular pyramid made of pasteboard to be unfolded, it is obvious that the surface of its

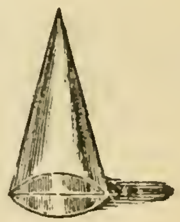




sides will be equal to that of C A B D E, which is composed of three equal triangles, but the area of C B D is C I $\times \frac{1}{2}$ B D therefore, the area of the three faces is C I $\times \frac{1}{2}$ (A B + B D + D E) = C I $\times \frac{1}{2} p$, p being the perimeter ; hence C I $\times \frac{1}{2} p$, together with the area of the base, is the whole surface. (Page 120.)

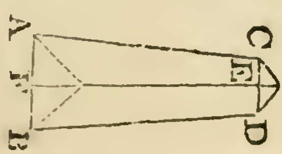
102. If a circular sector be described on paper, so that its radius shall be equal to the side of the cone, and its arc equal to the circumference of the base, this sector can be rolled round the cone, so as to cover it exactly ; but the area of





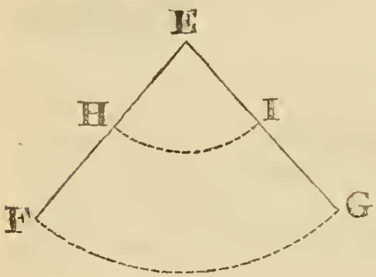
this sector is found by multiplying the radius of the sector by half the arc ; therefore, the convex surface of the cone is found by multiplying the slant height by half the circumference of the base, which, with the area of the base, is the whole surface. (Page 121.)

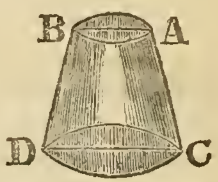
103. Let A B D C be one of the faces of the frustum ; E F, which joins the middle of A B and C D, is the slant height. Now, it is obvious that the ends being regular polygons, the upright surface will consist of as many trape-



zoids, each equal to A B D C, as there are sides in the polygon, the common height being E F; but the area of the face A B C D is $\frac{AB + CD}{2} \times EF$; therefore, the area of the whole upright surface is $\frac{P + p}{2} \times EF$. (P and p being the perimeters of the two ends of the frustum,) to which add the areas of both ends, for the entire surface. (Page 122.)

104. If a part of the sector E F G, viz. H F G I, having H F = B D, be rolled round the frustum A B D C, so as to cover it exactly, it is evident that the area of the envelope





H F G I will be equal to the convex surface of the cone A B D C. But the area of the envelope is $\frac{HI + FG}{2} \times FH$, and H I is equal to the perimeter of the less end A B of the cone, and F G equal the perimeter of the greater base C D; therefore, $\frac{P + p}{2} \times BD$ is the convex surface of the cone, P being the perimeter of the greater base, and

p that of the less ; therefore, $\frac{P + p}{2} \times B\,D$, together with the areas of both ends, will be the entire surface. (Page 123.)

By mistake, reference is made to Demonstration 165, in page 124, and the next demonstration is 107, instead of 105.

107. It is proved in Dr. Lardner's *Euclid*, Prop. X. Book IV. Solid Geometry, that the surface of a sphere is equal to that of the circumscribed cylinder ; but the cylindrical surface is equal to the circumference of its base, which is equal to that of the sphere, multiplied by its altitude, which is equal to a diameter of the sphere. Therefore, the surface of the sphere is equal to its circumference, multiplied by its diameter. (Page 125.)

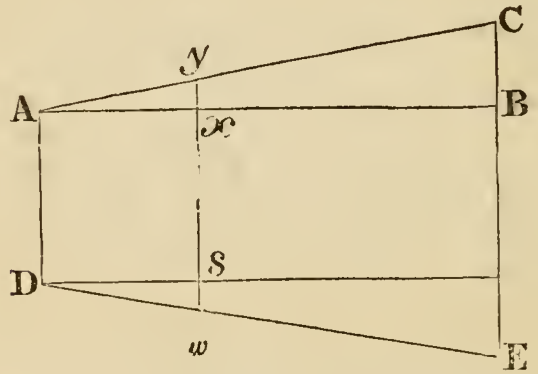
108. It is proved in Dr. Lardner's *Euclid*, Prop. XI. Book IV. Solid Geometry, that any plane intersecting a sphere and its circumscribing cylinder, parallel to the base of the cylinder, divides the spherical and cylindrical surfaces into parts which are equal to each other. Therefore, if two such planes be drawn, the spherical and cylindrical surfaces which they include will be the difference between the equal spherical and cylindrical surfaces which they cut off towards either of the bases of the cylinder, and, therefore, those differences are equal.

But the surface of the cylindrical segment is found by multiplying its circumference, which is equal to that of the sphere, by its length, which is equal either to the distance between the parallel planes, or to the height of the spherical segment: hence, the reason of the rule. (Page 126.)

109. The envelope of a cylinder is a parallelogram whose sides are evidently the height and circumference of the cylinder; therefore, the area of such a parallelogram is equal to the convex surface of the cylinder, to which the area of the two ends being added, the sum will give the entire surface of the solid. (Page 126.)

110. See Figure Problem XXVIII. Sec. IV. A C + C D $= m\,n$; then, the mean length of the cylinder is (A C + C D) $\times$ 3·1416; but the circumference of a section A C is A C $\times$ 3·1416; then, by the last Problem the surface is (A C + C D) $\times$ A C $\times$ 3·1416 $\times$ 3·1416 = (A C + C D) $\times$ A C $\times$ 9·8696, which is the rule. (Page 127.)

111. Let the area of the part required to be cut off be a. By similar triangles, we have A B : B C :: A x : $x\,y$ = $\dfrac{\text{B C} \times \text{A}\,x}{\text{A B}}$. But A x $\times$ $x\,y$ = the areas of the triangles.



A x y and D w s; therefore, $\dfrac{\text{B C} \times \text{A}\,x}{\text{A B}} \times \text{A}\,x =$

$\frac{B C \times A x^2}{A B}$ is the sum of the areas of the triangles A x y and D w s. Now, the rectangle D s x A, together with the adjacent triangles, is equal to a; therefore, we have

$\frac{B C \times A x^2}{A B} + A D \times A x = a$, a quadratic equation, which solved gives $A x = \frac{1}{2 B C} (A B^2 \times A D^2 + 4 B C \times A B \times a)^{\frac{1}{2}} - A B \times A D$. (Page 137.)

112. Because the surface of a sphere is equal to the curved surface of its circumscribed cylinder; but the curved surface of a cylinder whose diameter is D, and height $\frac{D}{2}$ has been shown to be $3 \cdot 1416 \times D \times \frac{D}{2} = 1 \cdot 5708 D^2$. Hence, the reason of the rule. (Page 169.)

113. It is known by experiment that an iron ball of 4 inches in diameter weighs 9 lbs.; and the weights of bodies composed of the same materials being as their quantities of matter; that is, as the cubes of their diameters. It will be as 4^3 (64) : the cube of any other ball :: 9 lbs. : the weight required, which affords the rule. (Page 188.)

114. It is found by experiment that a leaden ball of $4\frac{1}{4}$ inches diameter weighs 17 lbs. But the cube of $4\frac{1}{4}$ is to 17 nearly as 9 to 2. Hence, and from the proportion, that similar

solids are as the cubes of their diameters, the rule is obvious. (Page 189.)

115. The reason of this rule may be easily derived from Problem I. For, by deducting the weight sufficient to fill the cavity, from the capacity of the external surface, the remainder will express the weight of the shell. Let d and D express the internal and external diameters; then, their weights will be $d^3 \times \frac{9}{64}$, and $D^3 \times \frac{9}{64}$ $\therefore$ their difference, viz. $(D^3 - d^3) \times \frac{9}{64}$, will give the weight of the shell. (Page 190.)

116. By experiment it is found that 1 lb. avoirdupois weight of gunpowder contains 31·06 cubic inches. And putting d for the internal diameter, the capacity of the shell is $d^3 \times \cdot 5236$. Hence, $31 \cdot 06 : d^3 \times \cdot 5236 :: 1$ lb. : the weight required in pounds; that is, $d^3 \times \cdot 5236 \div 31 \cdot 06 = \frac{d^3}{59 \cdot 32}$ which is the rule, according to the note. The rule in the text may be explained in a similar manner. (Page 191.)

117. Let l, b and d, represent the length, breadth, and depth, respectively; then, the content will be l, b, d; hence, as in the last, it will be as $31 \cdot 06 : l\,b\,d ::$ 1lb. : the weight required; that is, $\frac{l\,b\,d}{31 \cdot 06} = l\,b\,d \times \cdot 0322$, which is the rule. (Page 192.)

118. Put d equal the diameter of the cylinder, and l for the length, then, its content is $d^2\ l \times \cdot 7854$; then, as in the two last, $31 \cdot 06 : d^2\ l \times \cdot 7854 :: 1$ lb. : weight, that is, $d^2\ l \times \cdot 7854 \div 31 \cdot 06 = d^2\ l \div 40$ nearly, which is the rule. (Page 192.)

119. Retaining the same notation as in the last, and putting $w =$ the weight, we have $w \frac{d^2\ l}{40}$; then, $40\ w = d^2\ l$; divide both sides of the equation by d^2, and we get $= \frac{40}{{}^2}$ which is the rule. (Page 193.)

Note.—The foregoing rules only approximate the truth.

120. The reason of this rule is derived from the method for finding the sum of a triangular progression. Thus, in a triangular pile it is obvious that each course of balls is in the shape of a triangle. The pile has only one ball on the top ; this ball rests upon three balls, which form the second row ; and these three balls rest upon 6 balls, which form the third row ; and these 6 balls rest upon 10 balls, and so on. Then, the sum of all these balls is equal to $1 + 3 + 6 + 10 +$ &c. to n terms, n being the number of courses. But this series is $= 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) +$ &c. ...... to n terms. The nth term of this series is $= \frac{n. (n + 1)}{2}$, and the last term but one is $= \frac{n. (n - 1)}{2}$, &c. and the sum of all this expression is $= \frac{n\ (n + 1\ (n + 2)}{6}$ which is the rule. (Page 194.)

121. The top of the pile is one ball which rests upon 4 balls ; and these 4 rest upon 9 balls ; and these rest upon 16, and so on ; all the courses then form a progression, such as $1^2 + 2^2 + 3^2 + 4^2 + 5^2 +$ &c. ...... n^2. This progression is formed of the squares of the natural numbers, 1, 2, 3, 4, 5, &c. to n terms, the sum of which is $=$

$\frac{n\,(n+1)\,(2\,n+1)}{6}$, as shown in most books of Algebra, which affords the rule. (Page 195.)

122. The uppermost row consists of one row of balls; this row is supported by a double row, the length of which is one ball more than are contained in the uppermost row, and the breadth one ball more than the breadth of the uppermost row. The third rectangular course is 2 balls in length and 2 in breadth more than the highest row contains, and so on; therefore, if $r =$ the number of balls in the highest row; $2\,(r+1) =$ the number of balls in the second course; $3\,(r+2) =$ the number in the third row, &c. Hence the number of balls in the whole pile is $r + 2\,(r+1) + 3\,(r+2) +$ &c. $+ n\,(r + n - 1)$ where $n =$ the number of balls in the breadth of the bottom course. But the expression

$$r + 2\,(r+1) + 3\,(r+2) + \&c. \ldots\ldots n\,(r+n-1) = r + 2\,r + 3\,r + \&c. \ldots\ldots n\,r$$

$$+ 2 + 6 + 12 + \&c.\, n\,(n-1) = \frac{n\,(n+1)}{2}.\; r + \frac{n\,(n-1)\,(n+1)}{3}.$$

Now put $m =$ the number of balls in the length of the base; then $m - n = r - 1$, or $r = m - n + 1$; therefore, substituting $m - n + 1$ for r, we get the number of balls $=$

$$\frac{r\,(n+1)}{2}\,(m - n + 1) + \frac{n\,(n+1)\,(n-1)}{3}$$

$$= n\,(n+1)\,\left\{\frac{m-n+1}{2} + \frac{n-1}{3}\right. = \frac{n\,(n+1)\,(3\,m - n + 1)}{6},$$

which is the algebraic expression for the rule. (Page 196.)

THE END.

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